RESTRICTION OF TEST IDEALS TO HYPERSURFACES

by

Andrew Bydlon

A dissertation submitted to the faculty of
The University of Utah
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

Department of Mathematics
The University of Utah
August 2017

The University of Utah Graduate School

STATEMENT OF DISSERTATION APPROVAL

The dissertation of **Andrew Bydlon**

has been approved by the following supervisory committee members:

Karl Schwede ,	Chair(s)	**28 Apr 2017**	Date Approved
Anurag Singh ,	Member	**2 May 2017**	Date Approved
Christopher Hacon ,	Member	**2 May 2017**	Date Approved
Tommaso De Fernex ,	Member	**28 Apr 2017**	Date Approved
Wen-Ching Winnie Li ,	Member	**28 Apr 2017**	Date Approved

by **Peter Trapa** , Chair/Dean of

the Department/College/School of **Mathematics**

and by **David B. Kieda** , Dean of The Graduate School.

ABSTRACT

In positive characteristic algebraic geometry and commutative algebra, one of the most fundamental invariants of a variety X, or more generally a pair of a variety X with a divisor Δ, is the test ideal $\tau(X, \Delta)$. Larger test ideals imply that the singularities of X and Δ are mild, while smaller test ideals imply more severe singularities.

In characteristic zero, the notion of the multiplier ideal $\mathcal{J}(X, \Delta)$ serves a similar purpose. In addition, the restriction theorem implies that the singularities of a hypersurface H inside X are worse than that of X. However, for most choices of such a restriction, the severity of the singularities are unchanged. In this work, it is demonstrated that in positive characteristic, the corresponding statement is false. In particular, there is a large class of examples for which almost every hypersurface has distinctly worse singularities than the ambient variety.

For my parents, Karen and Robert.

CONTENTS

CHAPTER 1

INTRODUCTION AND CHARACTERISTIC P PRELIMINARIES

In positive characteristic algebraic geometry and commutative algebra, the Frobenius morphiam is one of the most important tools that allows for one to mitigate the problem of p not being invertible. In this section, I demonstrate the relationship between the local geometry of a scheme or variety and implications of the Frobenius. In particular, I include some classical results about determining regularity, Cohen-Macaulayness, and normality via studying $F^e_* R$.

1.1 The Frobenius Morphism

Throughout, it will be assumed that R is a Noetherian ring of equal characteristic $p > 0$. This is equivalent to saying that R contains a field K with the property that $p > 0$. These assumptions yield a homomorphism

$$F : R \to R : r \mapsto r^p$$

called the Frobenius. It is natural to iterate this morphism and write $F^e = F \circ \overset{\text{e-times}}{\cdots} \circ F$. If X is a scheme over a field of positive characteristic, the canonical nature of the Frobenius allows one to extend the Frobenius from an affine cover to a morphism of X, called the absolute Frobenius: $F : X \to X$, which is the identity on topological spaces but is exactly the p-power map on $\mathcal{O}_X(U)$.

Definition 1.1. *If M is an R-module, denote by $F_* M$ the module which as an abelian group is M, but has the R-module action induced by the Frobenius. Namely, if $m \in M$ and $r \in R$, then*[1] *the action of R is given by $r \cdot F_* m = F_* r^p m$.*

A ring R as above is called **F-finite** *if $F_* R$ is a finitely generated R-module.*

---

[1] For the purpose of keeping track of where m is being viewed, $F_* m$ is used to denote m in $F_* M$

Throughout this paper, it will be a running assumption that all rings are F-finite and Noetherian, so that all results about finitely generated modules over Noetherian rings are accessible.

In the study of singularities in positive characteristic, one of the most persistent R-modules that occurs is the module

$$\mathrm{Hom}_R(F^e_* R, R)$$

or for schemes X the sheaf

$$\mathscr{H}om_{\mathcal{O}_X}(F^e_* \mathcal{O}_X, \mathcal{O}_X)$$

which on local sections represents all the local p^{-e}-linear homomorphisms from $\mathcal{O}_X(U)$ to $\mathcal{O}_X(U)$: $\phi(r^{p^e}\alpha) = r\phi(\alpha)$.

If $R = S/I$ is a quotient of a regular ring S by an ideal I, then a prominent result of Fedder gives this module a well-understood structure:

Theorem 1.2. *If $S = K[x_1, \dots, x_n]$, K is perfect, and $R = S/I$, then*

$$\mathrm{Hom}_R(F^e_* R, R) \cong \left(F^e_*(I^{[p^e]} :_S I) / F^e_*(I^{[p^e]}) \right) \cdot \mathrm{Hom}_S(F^e_* S, S).$$

The proof will be given below as Proof 1.2.1 after some results about regular rings are stated. Another correspondence for normal varieties X is the relationship between $\mathrm{Hom}_{\mathcal{O}_X}(F^e_* \mathcal{O}_X, \mathcal{O}_X)$ and the global sections of the line bundle $\mathcal{O}_X\left((1-p^e)K_X\right)$, where K_X is a canonical divisor, shown as follows:

$$\begin{aligned}
&\mathscr{H}om_{\mathcal{O}_X}(F^e_* \mathcal{O}_X, \mathcal{O}_X) && \\
\cong\ &\mathscr{H}om_{\mathcal{O}_X}(F^e_*(\mathcal{O}_X) \otimes \mathcal{O}_X(K_X), \mathcal{O}_X(K_X)) && \text{Twist and untwist by } \mathcal{O}_X(K_X) \\
\cong\ &\mathscr{H}om_{\mathcal{O}_X}(F^e_*(\mathcal{O}_X(p^e K_X), \mathcal{O}_X(K_X)) && \text{The Projection Formula} \\
\cong\ &F^e_* \mathscr{H}om_{\mathcal{O}_X}(\mathcal{O}_X(p^e K_X), \mathcal{O}_X(K_X)) && \text{Grothendieck duality for finite maps} \\
\cong\ &F^e_* \mathscr{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X((1-p^e)K_X)) && \text{Twisting} \\
\cong\ &F^e_* \mathcal{O}_X((1-p^e)K_X). &&
\end{aligned}$$

Note that these twisting operations induce isomorphisms at each step due to reflexivity of the corresponding sheaves. Taking global sections, and using the fact that F^e_* doesn't change global sections, shows that

$$\mathrm{Hom}_{\mathcal{O}_X}(F^e_* \mathcal{O}_X, \mathcal{O}_X) \cong H^0(\mathcal{O}_X\left((1-p^e)K_X\right)).$$

In particular, this module yields a great deal of information about the geometry of the scheme X. In the next section, this idea is expanded upon.

1.2 Geometric Properties of Frobenius

The absolute Frobenius morphism has long been seen as a method of studying the geometric properties of a scheme. Below are some of the historical results that served as motivating forces in this realization.

1.2.1 Regularity

Regular rings, or nonsingular varieties, are the 'nicest' possible in many respects.

Definition 1.3. *Let $(R, \mathfrak{m})$ be a local ring. Then R is said to be regular if*

$$\dim(R) = \dim_{R/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2).$$

The right-hand side of the equality is also equal to the minimal number of generators of $\mathfrak{m}$.

More generally, a Noetherian ring R (or scheme) is called regular if all of its localizations $R_\mathfrak{p}$ are regular local rings.

Many of the rings of interest in algebraic geometry are quotients of such regular rings over a field K. One of the earliest motivation for using the Frobenius as a method to study the singularities of a scheme is as follows:

Theorem 1.4. *Let R be an F-finite reduce ring over a field of characteristic $p > 0$. Then R is a regular ring if and only if $F^e_* R$ is a flat R-module.*

The original proof of Kunz is a technical dive into the Lech independence and element chasing. A substantially more modern proof of the flat implies regular direction is illustrated here using results of Bhatt and Scholze on perfections of a ring.

Proof. First, assume R is a regular ring. Taking $\hat{R}$ to be the completion of R at the unique maximal ideal yields

$$\hat{R} \cong K[\![x_1, \ldots, x_n]\!] \hookrightarrow R' = K[\![x_1^{\frac{1}{p^e}}, \ldots, x_n^{\frac{1}{p^e}}]\!] \hookrightarrow F^e_* \hat{R} \cong F^e_*(\hat{R}) \cong K^{\frac{1}{p^e}}[\![x_1^{\frac{1}{p^e}}, \ldots, x_n^{\frac{1}{p^e}}]\!]$$

via Cohen's structure theorem. The first extension of rings is R-free, since we have a natural basis for R' over R given as

$$\{x_1^{\frac{\alpha_1}{p^e}} \cdots x_n^{\frac{\alpha_n}{p^e}} \; : \; 0 \leq \alpha_i < p^e\}.$$

Finally, $F_*^e R = K^{\frac{1}{p^e}} \otimes_K R'$, and any extension of fields is free. Therefore, the overall extension is free.

Now, suppose that $F_*^e R$ is a flat R-module. To use the results of Bhatt and Scholze, the notion of a perfection of a ring needs to be introduced:

Definition 1.5. *Let R be a reduced ring of characteristic $p > 0$. Define R^∞, the perfection of R, to be*

$$R^\infty = \varinjlim F_*^e R$$

where the right-hand side is the directed system of Frobenius morphisms.

Informally speaking, we are adjoining all the p^{th}-roots of R and forming a ring of them. This procedure affords us the following lemmas:

Lemma 1.6. *[1, Lemma 3.16,5.10] Given $f : S \to R_1$ abd $g : S \to R_2$ surjections of Noetherian rings of characteristic $p > 0$. Then*

$$\mathrm{Tor}_i^{S^\infty}(R_1^\infty, R_2^\infty) = 0 \; i > 0.$$

Lemma 1.7. *[1, Lemma 5.31] If R is a complete regular local ring, then R^∞ has finite global dimension.*

Thus the statement of interest is then equivalent to showing that if R is a complete local ring of characteristic $p > 0$, and R^∞ is a flat R-module, then R is regular. By Lemma 1.7, R^∞ has finite global dimension, and furthermore, $R \to R^\infty$ is a faithfully flat extension. Thus regularity of R is equivalent to proving that R has finite global dimension.

Let n be the global dimension of R^∞. Let M, N be finitely generated modules with $Ext_R^i(M, N) \neq 0$ for $i > n$. Then by faithful flatness, we have

$$Ext_R^i(M, N) \otimes_R R^\infty \cong Ext_{R^\infty}^i(M \otimes_R R^\infty, N \otimes_R R^\infty) \neq 0.$$

This contradicts the fact that R^∞ has global dimension n, and by faithful flatness proves that R is regular. $\square$

Now I return to the proof of Fedder's Criterion, e.g. Theorem 1.2, from above. First note that $\mathrm{Hom}_S(F_*^e S, S)$ is in fact an $F_*^e S$ module of rank 1, with generating morphism $\Phi_S : F_*^e S \to S$ given by its action on the basis described above as

$$\Phi_S(u_i x_1^{\alpha_1} \cdots x_n^{\alpha_n}) = \begin{cases} 1 & i = 1,\ \alpha_j = p^e - 1 \\ 0 & otherwise \end{cases}$$

Here, as above, u_i is representative of a basis of $F_*^e K$ over K, and $0 \leq \alpha_j < p^e$.

Proof. Let $\varphi \in \mathrm{Hom}_S(F_*^e S, S)$ and $F_*^e f \in F_*^e(I^{[p^e]} :_S I)$. Then $\varphi(F_*^e f \cdot I) \subset I$ by definition of the colon operator. Therefore, $\varphi(F_*^e f \cdot -)$ restricts to $\mathrm{Hom}_R(F_*^e R, R)$. Call this restriction ψ. This defines a map

$$F_*^e(I^{[p^e]} :_S I) \cdot \mathrm{Hom}_S(F *^e S, S) \to \mathrm{Hom}_R(F_*^e R, R).$$

In Theorem 1.4, it is shown that $F_*^e S$ is a flat S-module, and hence projective. Therefore, any map $\psi : F_*^e S \to R$ factors through S:

$$\begin{tikzcd} F_*^e S \arrow[r, dashed] \arrow[d] \arrow[dr, "\psi"] & S \arrow[d] \\ F_*^e R \arrow[r] & R \end{tikzcd}$$

Thus the map written above is surjective. It goes to show that the kernel is $I \cdot \mathrm{Hom}(F_*^e S, S) = F_*^e I^{[p^e]}$. It is clear that it is a subset of K, the kernel. Suppose φ acts as 0 upon restriction to R. Then $\varphi = \overline{\Phi}_S(F_*^e f \cdot -)$ for some $f \in F_*^e(I^{[p^e]} : I)$ as above. Acting as 0 on R is equivalent to $\Phi_S(F_*^e f \cdot u_i \boldsymbol{x}^{\boldsymbol{\alpha}}) \subset I$ for each element of the basis of $F_* S$ over S, which when writing

$$F_*^e f = \sum_{i,\boldsymbol{\alpha}} f_{i,\boldsymbol{\alpha}} F_*^e u_i \boldsymbol{x}^{\boldsymbol{\alpha}} = \sum_{i,\boldsymbol{\alpha}} F_*^e f_{i,\boldsymbol{\alpha}}^{p^e} u_i \boldsymbol{x}^{\boldsymbol{\alpha}}$$

yields $\mathrm{im}(\varphi) = \langle f_{i,\boldsymbol{\alpha}} \rangle \subset I$, or equivalently viewed in $F_*^e S$, $f_{i,\boldsymbol{\alpha}} \in F_*^e I^{[p^e]} = I F_*^e S$. Thus, in particular, $f \in F_*^e I^{[p^e]}$, which shows that $F_*^e I^{[p^e]} \supset K$, and completes the proof. $\square$

1.2.2 Sharply F-split Rings

Though regularity is the most ideal condition for rings and schemes, weakenings of this condition are often encountered in practice and yield extremely interesting geometry as a result. In the previous section, it was established that in the case of a regular ring,

F^e_*R is a flat R-module. Thus, if we localize at a maximal ideal $\mathfrak{m}$ (noting that localizing at any multiplicatively closed set commutes with F^e_*), we get that $F^e_*R_\mathfrak{m}$ is a free $R_\mathfrak{m}$-module of some rank. Thus $\mathrm{Hom}_{R_\mathfrak{m}}(F^e_*R_\mathfrak{m}, R_\mathfrak{m}) \cong \mathrm{Hom}_{R_\mathfrak{m}}(R_\mathfrak{m}^{\oplus n}, R_\mathfrak{m}) \cong R_\mathfrak{m}^{\oplus n}$. In particular, the evaluation morphism $\mathrm{Hom}(F^e_*R, R) \otimes_R F^e_*R \to R : \phi \otimes r \mapsto \phi(F^e_*r)$ is surjective. Thus we can easily weaken the condition of regularity to a new notion, called F-split.

Definition 1.8. *A ring R of characteristic $p > 0$ is said to be F-split if the evaluation map*

$$ev : \mathrm{Hom}(F^e_*R, R) \otimes_R F^e_*R \to R : \phi \otimes r \mapsto \phi(F^e_*r)$$

is surjective.

This is roughly equivalent to the characteristic 0 notion of R being log canonical. It is called F-split since this implies immediately that there is some $\phi \in \mathrm{Hom}(F^e_*R, R)$ splitting $R \to F^e_*R$: $\phi \circ F = \mathrm{Id}_R$ For schemes X, there is a local and global (called globally F-split) version of this notion which coincide for affine schemes. Some immediate consequences are listed below:

Lemma 1.9. *If a Noetherian ring R is sharply F-split, then R is reduced and weakly normal.*

Definition 1.10. *A scheme X is called weakly normal if every finite, birational, bijective morphism $f : Y \to X$ with inseparable residue field extensions is an isomorphism. It is called normal if the same condition holds without the bijective assumption. An equivalent characterization of weakly normal for integral domains is as follows: An integral domain R is called weakly normal if for every $x \in K(R)$, the fraction field of R, with $x^p \in R$, $x \in R$.*

Proof: If R is F-split, then $R \to F^e_*R$ splits, so it is necessarily injective, which is equivalent to saying $r^p \neq 0$ for any r. Thus R is reduced.

Let $x \in K(R)$ and $x^p \in R$. Let $\varphi : F^e_*R \to R$ be a splitting of Frobenius. One can consider

$$\varphi_K = \varphi \otimes \mathrm{Id}_K : F^e_*K(R) \cong F^e_*R \otimes_R K(R) \to K(R) \cong R \otimes_R K(R).$$

Note in particular that $\varphi_K(F^e_*1) = \varphi(F^e_*1) \otimes 1 = 1 \otimes 1 = 1$, and additionally $\varphi_K(\frac{x}{y}) = \frac{\varphi(x)}{y}$. Therefore, φ_K is a splitting of $K(R) \hookrightarrow F^e_*K(R)$. In particular, $\varphi_K(x^p) = \frac{\varphi(x^p)}{1} = \frac{x}{1} = x \in R$. Thus R is also weakly normal.

This notion of F-split can be further extended to pairs $(R, \mathfrak{a}^t)$:

Definition 1.11. *The data of a pair is a normal ring R and an ideal $\mathfrak{a}$, with a positive rational (or real) number t. $(R, \mathfrak{a}^t)$ is called sharply F-pure if the following natural evaluation map is surjective:*

$$F^e_* \mathfrak{a}^{\lceil t(p^e-1) \rceil} \cdot \mathrm{Hom}_R(F^e_* R, R) \otimes F^e_* R \to R.$$

A pair $(R, \mathfrak{a}^t)$ being sharply F-pure is a characteristic $p > 0$ restriction on how severe the singularities of R and $\mathfrak{a}^t$ can be simultaneously. Again, there is a global notion of this notion, globally F-regularity. I end this section with 2 examples of F-split varieties.

Example 1.12. *Let K be a perfect field of characteristic $p > 0$. Consider $R = K[x_1, \ldots, x_n] / \langle x_1 \cdots x_n \rangle$ to be the intersection of n coordinate hyperplanes. Then certainly R is not normal as it is clearly not R_1. By Fedder's Lemma (Theorem 1.2), we see that*

$$\mathrm{Hom}_R(F^e_* R, R) \cong F^e_* (I^{[p^e]} : I) / I \cdot \Phi_S.$$

Thus, since I is principally generated, $I^{[p^e]} : I = \langle x_1^{p^e-1} \cdots x_n^{p^e-1} \rangle$, so $\psi(-) = \overline{\Phi_S(F^e_* x_1^{p^e} \cdots x_n^{p^e} \cdot -)}$ is an $F^e_* R$-module generator of $\mathrm{Hom}_R(F^e_* R, R)$. In addition, $\psi(1) = 1$, so R is F-split. Thus R is an example of a variety which is weakly normal, but non-normal.

Example 1.13. *Let K be a perfect field of characteristic $p > 0$. Consider the ring $R = K[x, y, z] / \langle x^3 + y^3 + z^3 \rangle$, whose corresponding* Proj *is an elliptic curve if $p > 3$. Then again, $\mathrm{Hom}_R(F^e_* R, R)$ is generated by $\overline{\Phi_S(F^e_* (x^3 + y^3 + z^3)^{p-1} \cdot -)}$. When $p \equiv 1 \pmod 3$, then $i = \frac{p-1}{3}$ is an integer. Thus*

$$(x^3 + y^3 + z^3)^{p-1} = \binom{p-1}{\frac{p-1}{3}, \frac{p-1}{3}, \frac{p-1}{3}} x^{p-1} y^{p-1} z^{p-1} + g.$$

where $g \in \langle x^p, y^p, z^p \rangle$. Therefore, it is easy to check that R is F-split by applying Φ_S.

If $p \equiv 2 \pmod 3$, or $p = 3$, then there are no 3 positive integers (i_1, i_2, i_3) with the property that $3i_j < p$ and $i_1 + i_2 + i_3 = p - 1$, so a similar analysis demonstrates that the image $\overline{\Phi_S(F^e_* (x^3 + y^3 + z^3)^{p-1} \cdot -))}$ lands inside $\langle x, y, z \rangle$, and is thus not F-split.

The case $p \equiv 1 \pmod 3$ is the case where $\mathrm{Proj}(R)$ *is an ordinary elliptic curve, whereas $p \equiv 2 \pmod 3$ is either a singular curve or a supersingular elliptic curve. This relationship between ordinary and F-split varieties holds in broad generality (even for abelian varieties, [15]).*

1.2.3 F-regular Rings

In this section, I describe a strengthening of F-split varieties, which is the natural analogue of being Kawamata Log Terminal in characteristic 0.

Definition 1.14. *A Noetherian ring R of characteristic $p > 0$ is called strongly F-regular if for every $c \in R$ a non-zero divisor, there exists $e > 0$ and $\psi \in \mathrm{Hom}_R(F^e_* R, R)$ such that $\psi(F^e_* c) = 1$. Equivalently, for any non-zero divisor c,*

$$R \to F^e_* R \xhookrightarrow{\cdot F^e_* c} F^e_* R.$$

A pair $(R, \mathfrak{a}^t)$ is said to to be strongly F-regular if for every $c \neq 0$, $\exists \psi \in \mathrm{Hom}_R(F^e_* R, R)$ such that $\psi(F^e_*(c\mathfrak{a}^{\lceil t(p^e-1)\rceil})) = R$.

Again, this is a local phenomenon, and there is a global analog for schemes called globally F-regular. Letting $c = 1$, we see that any strongly F-regular ring is also F-split. Additionally, it is easy to see that any regular ring is strongly F-regular.

Example 1.15. *Consider the ring $S = K[x_1, \ldots, x_n]$ (or $\hat{S} = K[\![x_1, \ldots, x_n]\!]$, or any localization thereof), where K is an F-finite field. Let $f \in S$, and $\deg(f) = d$. Choose $e > 0$ sufficiently large such that $p^e > d$. Then in the basis of $F^e_* S$ over S, $F^e_* f$ has coefficients in K, for degree reasons. Projecting from a non-zero summand $S \cdot u_i \boldsymbol{x}^{\boldsymbol{\alpha}}$ of $F^e_* S$ maps $F^e_* f$ to $c_{i,\alpha}$, the coefficient of $F^e_* u_i x^\alpha$ for $F^e_* f$. Thus R is strongly F-regular. Additionally, this easily produces examples where $e = 1$ is not sufficient, as was the case in checking F-splittings.*

The next two propositions demonstrate that a ring R being strongly F-regular has substantial consequences for the geometry of $\mathrm{Spec}(R)$.

Proposition 1.16. *Suppose R is a strongly F-regular domain. Then R is normal.*

Proof. Let $\mathfrak{c} = \mathrm{Ann}_R(R^N/R)$ be the conductor ideal of R (the largest ideal of R which is simultaneously an ideal of R^N). Then R is normal if and only if $\mathfrak{c} = R$. So suppose by contradiction that this is not the case, namely $1 \notin \mathfrak{c}$. Let $0 \neq c \in \mathfrak{c}$ and $\psi \in \mathrm{Hom}_R(F^e_* R, R)$ be such that $\psi(F^e_* c) = 1$. ψ can be extended to the fraction field, so in particular, $\psi^N : F^e_* R^N \to K(R)$. Additionally, one can check $\mathrm{im}(\psi^N) \subseteq R^N$ [5].

Consider $\alpha \in \psi(F^e_* \mathfrak{c})$. Then $\alpha = \psi(F^e_* x)$ for some $x \in \mathfrak{c}$ and for $r \in R^N$,

$$\alpha \cdot r = \psi^N(F^e_* x \cdot r^{p^e}) = \psi(F^e_* x \cdot r^{p^e}) \in R.$$

Since $x \cdot r^{p^e} \in R$ by defintion of the conductor. So $\psi(F^e_* x) \in \mathfrak{c}$, and thus $\psi(\mathfrak{c}) \subset \mathfrak{c}$. This contradicts the assumption that $\psi(F^e_* c) = 1$, which completes the proof. $\square$

Therefore, Example 1.12 is an example of a variety which is sharply F-split but **not** strongly F-regular.

Proposition 1.17. *Let R be a strongly F-regular Noetherian Domain. Then it is Cohen-Macaulay.*

Proof. We can further assume R is local. To show R is Cohen-Macaulay, it suffices to show that for each $i < \dim(R)$, $H^i_{\mathfrak{m}}(R) = 0$. Fix such an i. By a support argument, one can show that $\exists 0 \neq c \in R$ such that $c \cdot H^i_{\mathfrak{m}}(R) = 0$. But R is strongly F-regular, so $\exists \psi : R \to F^e_* R : 1 \mapsto F^e_* c$ split in the category of R-modules. Applying the left exact functor $H^i_{\mathfrak{m}}(-)$ yields

$$H^i_{\mathfrak{m}}(R) \to H^i_{\mathfrak{m}}(F^e_* R) \xhookrightarrow{\cdot c} H^i_{\mathfrak{m}}(F^e_* R)$$

which is also a split morphism. But by virtue of the fact that $F^e_* R$ and R are isomorphic as rings,

$$H^i_{\mathfrak{m}}(F^e_* R) \cong F^e_* H^i_{\mathfrak{m}^{[p^e]}}(R) \cong F^e_* H^i_{\sqrt{\mathfrak{m}^{[p^e]}}}(R) \cong F^e_* H^i_{\mathfrak{m}}(R).$$

So multiplication by c is the zero map, which is also injective. Thus $H^i_{\mathfrak{m}}(R) = 0$, and R is CM. $\square$

Next I introduce the invariant of interest in this paper, the test ideal.

1.3 The Test Ideal

Let R be a normal ring over a perfect field K of characteristic $p > 0$, let $\Delta \geq 0$ be a divisor on $\operatorname{Spec}(R)$ such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier, and let $\mathfrak{a}$ be an ideal with $t > 0$. This is the data of a triple $(X, \Delta, \mathfrak{a}^t)$. The (big) test ideal associated to this data, denoted $\tau(X, \Delta, \mathfrak{a}^t)$, is defined as follows:

Definition 1.18. *$\tau(R, , \Delta, \mathfrak{a}^t)$ is the smallest non-zero ideal J of R such that for every $e > 0$ and every R-linear homomorphism $\phi : F^e_* R(\lceil (p^e - 1)\Delta \rceil \to R$, J is $\mathfrak{a}^t$-compatible with ϕ in the following sense:*

$$\phi\left(F^e_*(J \cdot \mathfrak{a}^{\lceil t(p^e-1) \rceil})\right) \subseteq J.$$

This is a local definition, so it can easily be extended to the case of triples of schemes over K: $(X, \Delta, \mathfrak{a}^t)$. Either or both of Δ and $\mathfrak{a}^t$ are often omitted, meaning that either $\Delta = 0$ or $\mathfrak{a} = R$.

Using this definition, it is not obvious that such a *smallest* ideal could exist, or even moreso how it could be computed in practice. The work of Hochster and Huneke on test elements mitigates this issue:

Theorem 1.19. *[13] Let R be an F-finite normal domain essentially of finite type over a field K. Given $\phi \in \operatorname{Hom}_R(F^e_* R, R)$, then $\exists c \neq 0$ in R such that for any $d \neq 0$, one can find $f \gg 0$ such that $c \in \phi^n(F^e_* \langle d \rangle)$. Here ϕ^n is an abuse of notation for $\phi \circ F^e_* \phi \circ \cdots \circ F^{(f-1)e}_* \phi : F^{fe}_* R \to R$.*

It should be noted that often times, c is not terribly difficult to find. Such a c produces the following result:

Theorem 1.20. *Let c be an element as in Theorem 1.19. Then*

$$\tau(R) = \sum_{e \geq 0} \sum_{\psi} \psi(F^e_* \langle c \rangle)$$

where ψ runs through $\operatorname{Hom}_R(F^e_* R, R)$.

Proof. First, I show that $\tau(R) \subset \sum_{e \geq 0} \sum_{\psi} \psi(F^e_* \langle c \rangle)$. It is enough to show that $\sum_{e \geq 0} \sum_{\psi} \psi(F^e_* \langle c \rangle)$ is φ-compatible, as $\tau(R)$ is the smallest such ideal.

$$\varphi \left(F^a_* \sum_{e \geq 0} \sum_{\psi} \psi(F^e_* \langle c \rangle) \right) = \sum_{e \geq 0} \sum_{\psi} \varphi \left(F^a_* \psi(F^e_* \langle c \rangle) \right) \subseteq \sum_{e \geq 0} \sum_{\psi} \psi(F^e_* \langle c \rangle)$$

as $\phi \circ F^a_* \psi$ is just another Hom already represented in the sum.

Now for the reverse inclusion. Note that since $\tau(R)$ is in particular non-zero, it contains some $d \neq 0$. Therefore, the definition of the test element above implies $c \in \psi^n(F^e_* d \cdot R)$. Since $\tau(R)$ is in particular ψ^n-compatible, this implies $c \in \tau(R)$. But this subsequently implies that $\psi(F^e_* \langle c \rangle) \subseteq \tau(R)$ for every $\psi \in \operatorname{Hom}_R(F^e_* R, R)$ and every $e \geq 0$. □

Also, note that because R is a Noetherian ring, this sum is a finite sum, so finitely many e suffice. This can further be extended to the case of triples $(R, \Delta, \mathfrak{a}^t)$ with a generalized test element $c \in \tau(R; \psi)$, the smallest $\psi : F^e_* R(\lceil t(p^e - 1)\Delta \rceil)$-compatible ideal.

Additionally, we have the relatively easy implication from the previous subsection:

$$(R, \mathfrak{a}^t) \text{ is strongly F-regular} \iff \tau(R, \mathfrak{a}^t) = R.$$

Since strongly F-regular singularities are very mild, the following ideology is often applicable: Test ideals which are large (close to R) means that $(R, \Delta, \mathfrak{a}^t)$ mild singularities, and smaller test ideals (closer to $\langle 0 \rangle$) have more severe singularities.

Example 1.21. *Consider $R = K[x,y,z]$ with $char(K) = p \neq 2$, and $\mathfrak{a} = \langle x^2y - z^2 \rangle$. Therefore,*

$$\tau(R, \mathfrak{a}^t) = \sum_e \Phi_S^e(F_*^e \langle c \cdot (x^2y - z^2)^{\lceil t(p^e-1) \rceil} \rangle).$$

Note that if $t = \lfloor t \rfloor + \{t\}$, then $\tau(R, \mathfrak{a}^t) = \mathfrak{a}^t \cdot \tau(R, \mathfrak{a}^{\{t\}})$. So in particular, if t is an integer, then $\tau(R, \mathfrak{a}^t) = \mathfrak{a}^t$.

So it suffices to consider the test ideal for $0 \leq t < 1$: Let c be any test element. Then

$$\tau(R, \mathfrak{a}^t) = \sum_{e \geq 0} \sum_{\Phi} \Phi(F_*^e c \mathfrak{a}^{\lceil t(p^e-1) \rceil}).$$

Since $t < 1$, we can consider $e \gg 0$ such that $c \cdot (x^2 - yz^2)^{\lceil t(p^e-1) \rceil} \notin \mathfrak{m}^{[p^e]}$. Therefore, $\tau(R, \mathfrak{a}^t) = R$ by Nakayama's Lemma. Thus, the test ideal jumps at $1, 2, 3, \ldots$.

The test ideal $\tau(R, \mathfrak{a}^t)$ measures the singularities of R and $\mathfrak{a}^t$ ($V(\mathfrak{a})$ with a coefficient of t) simultaneously. In fact, $V(\tau(R))$ is exactly the locus of $\operatorname{Spec}(R)$ which is not strongly F-regular. The following theorem demonstrates an additional instance of this.

Theorem 1.22. *[8] Suppose R is a F-split ring. Then $R/\tau(R)$ is also F-split.*

Proof. Let $\psi : F_*^e R \to R$. Since $\tau(R)$ is in particular ψ compatible, we have

$$\psi(F_*^e \tau(R)) \subseteq \tau(R).$$

Therefore, ψ descends to a map $\overline{\psi} : F_*^e R/\tau(R) \to R/\tau(R)$, and $\overline{\psi(1)} = 1$, so $\overline{\psi}$ is a splitting of $R/\tau(R)$. $\square$

1.4 Bertini Theorems

Bertini theorems are a general term, derived from the original theorem of Bertini on smoothness over algebraically closed fields, which regards properties of a projective variety which are kept intact for an open dense set of its subvarieties of the form $H \cap X$ for a hypersurface $H \subset \mathbb{P}^n$. They are in particular very powerful as a technique to prove results on higher dimensional varieties by considering their lower dimensional pieces. The original result of Bertini is stated here for historical purposes:

Theorem 1.23. *Let $X \subset \mathbb{P}_K^n$ be a smooth projective variety over an algebraic closed field K. Then for an open dense subset $U \subset (\mathbb{P}^n)^\vee$, and all $H \in U$, the scheme theoretic intersection $X \cap H$ is* *smooth. If* $\dim(X) \geq 2$*, then* $\exists U$ *open with the previous property, and we may assume* $X \cap H$ *is irreducible.*

This motivated many theorems of nice properties $\mathcal{P}$ with this Bertini-type property. In [7], an axiomatic system was developed to show that a given property behaves well under restriction to a general hyperplane:

Theorem 1.24. *Let* X *be a scheme of finite type over an algebraically closed field* K*, and let* $\phi : X \to \mathbb{P}^n_K$ *be a morphism with separably generated residue field extensions. Suppose* $\mathcal{P}$ *is a local property of locally Noetherian schemes such that*

1. *whenever* $\varphi : Y \to Z$ *is a flat morphism with regular fibers and* Z *is* $\mathcal{P}$*, then* Y *is* $\mathcal{P}$*.*
2. *let* $\varphi : Y \to S$ *be a morphism of finite type, where* Y *is excellent and* S *is integral with generic point* η*. Then if* Y_η *is geometrically* $\mathcal{P}$*, then there exists an open neighborhood* U *of* η *such that* Y_s *is geometrically* $\mathcal{P}$ *for each* $s \in U$

Then there exists a nonempty open set $U \subset (P^n_K)^*$ *such that for all* $H \in U$*,* $\phi^{-1}(H)$ *has the property* $\mathcal{P}$*.*

This led to several other properties of varieties, including Theorem 2.6 of Schwede and Zhang where it was demonstrated that $\mathcal{P}$ =(strong F-regularity) and $\mathcal{P}$ =(sharp F-split) have the properties of Theorem 1.24.

CHAPTER 2

MOTIVATION AND CHARACTERISTIC 0

Complex algebraic geometry is amongst the most well-studied and understood regions of algebraic geometry. Many desirable theorems such as Kawamata-Viehweg vanishing and the Hodge decomposition theorems are known to hold, but do not hold in positive or mixed characteristic. However, often times, one can pass from the complex setting to positive characteristic, by a method described below, to prove certain theorems in positive characteristic, or vice-versa. In this section, the multiplier ideal is discussed in depth, and many of the important results motivating a Bertini-like conjecture for test ideals are stated.

2.1 The Multiplier Ideal

Let $X \subseteq \mathbb{P}^n$ be a smooth quasi-projective variety over $\mathbb{C}$.

Definition 2.1. *Let $\Delta \geq 0$ be a $\mathbb{Q}$-divisor such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier, and let $\pi : X' \to X$ be a log resolution of Δ. The multiplier ideal sheaf is defined as*

$$\mathcal{J}(X, \Delta) = \pi_* \mathscr{O}_{X'}(\lceil K_{X'} - \pi^*(K_X + \Delta) \rceil).$$

This notion is independent of the log resolution chosen, and measures the singularities of the pair (X, Δ) in characteristic zero. Additionally, this is a local notion. A simple example of a cone over an elliptic curve is now shown:

Example 2.2. *Let $X = \mathrm{Spec}(\mathbb{C}[x, y, z]/(x^3 + y^3 - z^3)$, and let $\Delta = 0$. Consider the following diagram:*

$$\begin{array}{ccc} \tilde{X} & \longrightarrow & X \\ \downarrow & & \downarrow \\ Bl_0\mathbb{P}^3 & \xrightarrow{\pi} & \mathbb{P}^3 \end{array}$$

where $\tilde{X}$ is the embedded blowup of X at the origin. It is easy to check that π is a log resolution of the pair (X, Δ). Furthermore, one can compute

$$\mathcal{J}(X) = \pi_* \mathcal{O}_{X'}(\lceil K_{X'/X}) = \pi_* \mathcal{O}_{X'}(\lceil -E\Delta) \rceil) = \mathfrak{m}$$

where $\mathfrak{m} = \langle x, y, z \rangle$ is the ideal at the origin. So the multiplier ideal is detecting information about the singular (non-KLT) locus of (X, Δ).

2.2 The Restriction Theorem

The primary motivating theorem is as follows (see [14, 9.2.29 and 9.5.9]):

Theorem 2.3. *Let (X, Δ) be a pair, with X a smooth complex variety and $\Delta \geq 0$ a $\mathbb{Q}$-divisor. Let $|V|$ be a base-point free linear series on X. Then for a general choice of divisor $B \in |V|$, and any $0 \leq \epsilon < 1$,*

$$\mathcal{J}(X, \Delta) = \mathcal{J}(X, \Delta + \epsilon B).$$

In addition, for general $B \in |V|$,

$$\mathcal{J}(X, D) \otimes_{\mathcal{O}_X} \mathcal{O}_B \cong \mathcal{J}(B, D|_B).$$

So multiplier ideals behave as well as one could expect when taking general hyperplane sections. Moreover, as shown in the next section, multiplier ideals are intimately related to test ideals.

2.3 Reduction to Characteristic p

Let $X = \operatorname{Spec}(R)$ be a complex variety, with $R = \mathbb{C}[x_0, \ldots, x_n]/I$ with $D \geq 0$ an effective divisor. If $I = \langle f_1, \ldots, f_m \rangle$, then let A be the $\mathbb{Z}$-algebra generated by all of the coefficients of the f_i and the equations of a log resolution $\pi : \tilde{X} \to X$. Then X and π are defined over A in the sense that we can find $X_A \to A$ such that $X_A \otimes_A \mathbb{C} = X$.

Claim 2.4. *For each maximal ideal $\mathfrak{m} \subseteq A$, $A/\mathfrak{m}$ is a finite field.*

Proof. The only thing to check is that the field is finite. Note additionally that $k = A/\mathfrak{m}$ is a field of characteristic $p > 0$: Suppose (aiming for a contradiction) that k is a field of characteristic 0. Then $\mathbb{Q} \hookrightarrow k$. But k is a finitely generated $\mathbb{Z}$-algebra, since A is, whereas $\mathbb{Q}$ is not, a contradiction. So it only remains to show that k is finite. But A is finitely generated over $\mathbb{Z}$, so $A/\mathfrak{m}$ is finitely generated over $\mathbb{Z}/\pi^{-1}(\mathfrak{m}) = \mathbb{Z}/p$. This completes the proof. $\square$

Therefore, we can consider $X_{\mathfrak{m}}$ obtained by base change:

$$\begin{array}{ccc} X_{\mathfrak{m}} = X_A \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A/\mathfrak{m}) & \longrightarrow & X_A \\ \downarrow & & \downarrow \\ Spec(A/\mathfrak{m}) & \longrightarrow & Spec(A) \end{array}$$

The following theorem motivates the idea that many of the well-studied properties of multiplier ideals have a chance to descend to test ideals:

Theorem 2.5. *[18], cf. [17],[12],[10] Let X be a complex projective variety, and Δ a $\mathbb{Q}$-divisor on X with $K_X + \Delta$ a $\mathbb{Q}$-Cartier divisor. Then for a given model $X_A \to A$ of $X \to \mathbb{C}$, there is an open dense collection of closed points $\mathfrak{m} \in$ m-Spec(A) such that*

$$\mathcal{J}(X, D)_{\mathfrak{m}} = \tau(X_{\mathfrak{m}}, D_{\mathfrak{m}}).$$

In particular, it seems reasonable to ask if the same Bertini-type theorem holds for test ideals as well.

2.4 The Strongly F-regular Case

A particular case of this more general conjecture is known in the case of a strongly F-regular pair. This corresponds to the case of $\tau(X, \Delta) = \mathcal{O}_X$, and the theorem is restated here:

Theorem 2.6. *[16, Theorem 6.7] Suppose that X is a variety over an algebraically closed field k, let $\Delta \geq 0$ be a $\mathbb{Q}$-divisor on X, and let $\phi : X \to \mathbb{P}^n_k$ be a k-morphism with separably generated residue field extensions[1]. Suppose (X, Δ) is a strongly F-regular pair. Then for an open dense subset $U \subseteq (\mathbb{P}^n_k)^\vee$ and all $H \in U$, $\left(\phi^{-1}(H), \Delta|_{\phi^{-1}(H)}\right)$ is strongly F-regular.*

This theorem is proved using Theorem 1.24, and demonstrating that sharply F-pure pairs have the two properties listed in the theorem.

Furthermore, one can easily test low-dimensional hypersurfaces of a fixed degree over a finite field using Macaulay2 [9]. There is a script created by the author available as

[1]Note that one cannot expect the same statement to hold for a general member of an arbitrary basepoint free linear system.

[6], which produces a list of hyperplanes for which the restriction fails. See Appendix Section A.1. There is an additional piece of code which has been used to test *all* examples of hypersurfaces of degree smaller than 6 in characteristics $p \leq 13$ (as well as small Galois field extensions of $\mathbb{F}_p$) in $\mathbb{P}^n_{\mathbb{F}_p}$ for $n \leq 4$. This process yielded only examples for which the proportion of hyperplanes failing the conjectured restriction theorem was very small. Thus it can be expected that these hyperplanes could be included as a component of a closed set upon taking the algebraic closure of the underlying field, and thus there would be an open set for which restriction holds.

CHAPTER 3

MAIN RESULTS

In this section, it is demonstrated that the restriction of a test ideal to a hyperplane (or even hypersurface) in dimension greater than or equal to 3 is not in general the test ideal of the restriction. Moreover, this can be used to demonstrate that a natural extension of a result of Hochster and Huneke on strongly F-regular flat families cannot be extended to test ideals.

3.1 Computation of $\tau(\mathbb{P}^n, \Delta)$

Consider the pair $(\mathbb{A}^n_K, f^{\frac{1}{p}})$. In general, $f^{\frac{1}{p}}$ represents the divisor defined by f with coefficient $\frac{1}{p}$, and f is an element of the ring $S = K[x_1, \ldots, x_n]$. Assume K is an F-finite field with a basis for F_*K over K given as $\langle F_*u_1, \ldots, F_*u_m \rangle$. Let

$$F_*f = \sum_{\boldsymbol{\alpha},i} s_{\boldsymbol{\alpha},i} F_*u_i \boldsymbol{x}^{\boldsymbol{\alpha}}.$$

In general, $\boldsymbol{\alpha}$ will denote a multi-index in $\{0, 1, \ldots, p-1\}^{\oplus n}$, coming from the set of exponents occurring in the standard basis of monomials F_*S over S. Similarly, $\boldsymbol{x}^{\boldsymbol{\alpha}}$ is shorthand for $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. The index i will range from 1 to m, to indicate the elements of the basis of F_*K over K as above. In general, a boldface symbol will denote a multi-index. Also note that $s_{\boldsymbol{\alpha},i} \in S$ is viewed naturally as a subset of F_*S under the inclusion $S \hookrightarrow F_*S$: $x \mapsto xF_*1 = F_*x^p$. Let Φ_S denote the generator of $\mathrm{Hom}_S(F_*S, S)$ as an F_*S-module under premultiplication:

$$\Phi_S : F_*S \to S : F_*\boldsymbol{u_1}\boldsymbol{x}^{\boldsymbol{p-1}} \mapsto 1$$

and Φ_S sends all of the other elements of the standard basis $\langle F_*u_i x^{\alpha} \rangle$ to 0.

The test ideal $\tau(\mathbb{A}^n, f^{\frac{1}{p}})$ can then be described by the image of $\Phi_S(F_*f \cdot -)$ which can be computed explicitly as follows. This was written down as early as [11], but the proof following the notation outlined above is provided here for convenience:

Proposition 3.1. *[2]* $\tau(\mathbb{A}^n, f^{\frac{1}{p}}) = \Phi_S(\langle F_* f\rangle) = \langle s_{\alpha,i}\rangle$.

Proof. The first equality goes as follows: $\Phi_S^{\circ e}$ generates $\mathrm{Hom}_S(F_*^e S, S)$ for each $e > 0$. Therefore, for $c \in \tau(R, \psi)$,

$$\tau(\mathbb{A}^n, f^{\frac{1}{p}}) = \sum_{e \geq 0} \Phi_S(c \cdot \langle f\rangle^{p^{e-1}})$$

however, a result of Hara-Takagi states the following:

Theorem 3.2. *[11] If c is a test element for R (not the pair $(R, \mathfrak{a}^t)$), then*

$$\tau(R, \mathfrak{a}^t) = \sum_e \sum_\psi \psi(c\mathfrak{a}^{\lceil tp^e\rceil}).$$

So in particular,

$$\begin{aligned} \Phi_S^e\left(F_*^e\langle f^{\lceil \frac{1}{p}(p^e)\rceil}\rangle\right) &= \Phi_S^e\left(F_*^e\langle f^{\lceil tp^{e-1}\rceil}\rangle\right) \\ &\subseteq \Phi_S\left(F_*\langle f \cdot \Phi_S^{e-1}(F_*^{e-1}S\right) \subseteq \Phi_S(\langle F_* f\rangle). \end{aligned}$$

The other containment is exactly the $e = 1$ term of the sum.

Lemma 3.3. *[11, Corollary 2.4] If $(R, \mathfrak{m})$ is an F-finite local ring of characteristic $p > 0$, and the non-finitistic test ideal $\tau_b(R)$ agrees with $\tau(R)$, then if c is a test element of R, then it is a test element of $(R, \mathfrak{a}^t)$ for all $\mathfrak{a} \cap R^\circ \neq 0$ and rational numbers $t \geq 0$.*

So in particular, it only remains to show $\Phi_S(\langle F_* f\rangle) = \langle s_{\alpha,i}\rangle$. Consider an element g of $\Phi_S(\langle F_* f\rangle)$. Then S-linearity of Φ_S implies

$$g = \Phi_S(F_* h \cdot f) = \sum_{\boldsymbol{\alpha}, m} s_{\boldsymbol{\alpha},i}\Phi_S(F_*(u_i \cdot h \cdot \boldsymbol{x}^{\boldsymbol{\alpha}})) \subseteq \langle s_{\alpha,i}\rangle.$$

Similarly (with a clever choice of basis), considering the element $\Phi_S\left(F_*(u_i^{-1} \cdot \boldsymbol{x}^{\boldsymbol{p-1-\alpha}} \cdot f)\right) = s_{\alpha,i} \in \Phi_S(\langle F_* f\rangle)$ yields the reverse inclusion. □

Let H be a hyperplane section in $\mathbb{A}^n$ defined by $l = c_0^p + c_1^p x_1 + \ldots + c_n^p x_n$. Considering the pair $(\mathbb{A}_K^n, (f \cdot l)^{\frac{1}{p}})$ where we can conclude identically to the previous case that the corresponding test ideal $\tau(\mathbb{A}_K^n, (f \cdot l)^{\frac{1}{p}})$ is given as the image of $\Phi_S(F_*(f \cdot l \cdot S))$, or equivalently

$$\langle c_0 s_{\boldsymbol{\alpha},i} + \sum_{j=1,\ldots,n} c_j \cdot s_{\boldsymbol{\alpha}-1_j,i} \cdot x_j^{\delta_{\alpha_j,p-1}}\rangle.$$

Note here that δ is the dirac delta function, and $\boldsymbol{\alpha} - 1_j$ is taken $\pmod p$. Additionally, for each $\boldsymbol{\alpha}$ in the set $\{0, 1, \ldots, p-1\}^{\oplus n}$, there is a corresponding generator for $\tau(\mathbb{A}_K^n, (f \cdot l)^{\frac{1}{p}})$.

$\mathbf{1}_j$ is representative of a multi-index with 1 in position j and 0 elsewhere. So rephrasing the statement from Section 3.1, to disprove the direct analog of Theorem 2.3, it is enough to show that for a dense (or in the case of Theorem 3.4, a dense open subset) set of $H = V(l)$, the two ideals formed as the image of $\Phi_S(F_* f \cdot -)$ and $\Phi_S(F_* f \cdot l \cdot -)$ do not agree. This will be shown below as Theorem 3.4.

One can further consider the case of the test ideal sheaf $\tau(\mathbb{P}^n, f^{\frac{1}{p}})$, which on each element of the standard affine cover is $\tau(\mathbb{A}^n, f^{\frac{1}{p}}|_{x_i=1})$ since the big test ideal is defined locally. This is recorded in Corollary 3.8 and Theorem 3.23 for the sake of completeness.

Let $K = K^p$ be a perfect field, and $S = K[x_1, \ldots, x_n]$, and consider $\mathbb{A}^n = \operatorname{Spec}(S)$ in which case the trace map can be viewed as

$$\Phi_S : F_* S \to S.$$

Continuing the use of the multi-index notation $\boldsymbol{x}^{\boldsymbol{\alpha}} = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, Φ_S is defined by=

$$\Phi_S(\boldsymbol{x}^{\boldsymbol{\alpha}}) = \begin{cases} \boldsymbol{x}^{\frac{\boldsymbol{\alpha}-\boldsymbol{p}+\boldsymbol{1}}{p}} & p \mid \alpha_i + 1 \ \ \forall i \\ 0 & \text{otherwise} \end{cases}$$

3.2 Counterexamples to Restriction Theorems for Test Ideals

The best case scenario for extending Bertini theorems to characteristic $p > 0$ would be a direct generalization of the Bertini Theorem for Multiplier Ideals. That is to say that for I homogeneous, $R = S/I$, and Δ a $\mathbb{Q}$-Cartier Divisor on $X = \operatorname{Spec}(R)$, is it true that for $0 \leq \epsilon < 1$ and a general hyperplane H, the analogue of [14, 9.2.29] holds true?

$$\tau(R, \Delta) \stackrel{?}{=} \tau(R, \Delta + \epsilon H)$$

$$\tau(X, \Delta)|_H \otimes_{\mathscr{O}_X} \mathscr{O}_H \stackrel{?}{=} \tau(H, \Delta|_H).$$

3.2.1 Main Result

The following examples show that this is not the case. All counter-examples are worked out in the affine case $S = K[x_1, \ldots, x_n]$, but extend naturally to $\mathbb{P}^n$.

Theorem 3.4. *Suppose that $S = K[x_1, \ldots, x_n]$ is a polynomial ring over an infinite perfect field K of characteristic $p > 0$, with $n \geq 3$. Consider an element $f \in S$ of the form*

$$F_* f = f_{p-1} \cdot F_* \boldsymbol{x}^{\boldsymbol{p-1}} x_n^{p-1} + f_{p-2} \cdot F_* (\boldsymbol{x}^{\boldsymbol{p-1}} x_n^{p-2} + \ldots + f_0 \cdot F_* \boldsymbol{x}^{\boldsymbol{p-1}}$$

where $\boldsymbol{x^{p-1}} = x_1^{p-1} \cdots x_{n-1}^{p-1}$*, and the* f_i *are polynomials in* $K[x_1, \ldots, x_{n-1}]$ *with the additional independence property that for each* $i = 0, \ldots, p-1$*,*

$$f_i \notin \langle f_0, \ldots, f_{i-1} \rangle + \mathfrak{p} \cdot \tau(S, f^{\frac{1}{p}}) = \langle f_0, \ldots, f_{i-1}, x_j f_i, \ldots, x_j f_{p-1} \rangle \qquad (\star)$$

where $\mathfrak{p} = \langle x_1, \ldots, x_{n-1} \rangle$ *and* $j = 1, \ldots, n-1$*. Then there is an open dense subset of hyperplanes* $H = V(l)$ *such that*

$$\tau(S, f^{\frac{1}{p}}) \neq \tau(S, (l \cdot f)^{\frac{1}{p}}).$$

Proof. Following Proposition 3.1, one can conclude directly that $\tau(S, f^{\frac{1}{p}}) = \langle f_0, \ldots, f_{p-1} \rangle$ and giving the linear form l a presentation of $l = c_0^p + c_1^p x_1 + \ldots + c_n^p x_n$, we can conclude $\tau(S, (f \cdot l)^{\frac{1}{p}})$ is generated by elements of the form:

- $c_0 f_i + c_n f_{i-1}$ for $i = 1, \ldots, p-1$, the coefficient of $F_*(x_1 \cdots x_{n-1})^{p-1} x_n^i$.
- $c_0 f_0 + c_n x_n f_{p-1}$, the coefficient of $F_*(x_1 \cdots x_{n-1})^{p-1}$.
- $c_j x_j f_i$ for $i = 0, \ldots, p-1$, $j = 1, \ldots, n-1$, the coefficient of $F_*(x_1 \cdots \hat{x}_j \cdots x_{n-1})^{p-1} x_n^i$.

Restrict to considering only those l for which all of the c_j are non-zero[1], which will stand as the open condition in the theorem. To prove the claim, note that it is enough to show that $f_{p-1} \notin \tau(S, (f \cdot l)^{\frac{1}{p}})$, which by considering the generators of the first type above is equivalent to showing every $f_i \notin \tau(S, (f \cdot l)^{\frac{1}{p}})$. Note this will be used in the proof of Proposition 3.17. Suppose (aiming for a contradiction) that f_{p-1} has a presentation as an element of $\tau(S, (f \cdot l)^{\frac{1}{p}})$:

$$f_{p-1} = g_0 \cdot (c_0 f_0 + c_n x_n f_{p-1}) + \sum_{i \leq p-1} g_i \cdot (c_0 f_i + c_n f_{i-1}) + \sum_{i,j} h_{i,j} \cdot x_j f_i.$$

where $g_i, h_{i,j} \in S$ are some coefficients, and the indices i, j come from the list of generators above.

Claim 3.5. *Without loss of generality, we may assume that* $g_i \in K[x_n]$*.*

Proof. The idea is that one can take any monomial terms of the original presentation of g_i in $\langle x_1, \ldots, x_{n-1} \rangle$, and instead include them in some combination of the $h_{i,j}$.

---

[1] It suffices that $c_n \neq 0$

Explicitly, if each $g_i = g_{i,0} + x_1 g_{i,1} + \ldots + x_{n-1} g_{i,n-1}$ with $g_{i,0} \in K[x_n]$, then one can replace g_i with $g_{i,0}$, replace $h_{p-1,j}$ with $h_{p-1,j} + c_0 g_{p-1,j} + c_n x_n g_{0,j}$, and replace each $h_{i,j}$ with $h_{i,j} + c_0 g_{i,j} + c_n g_{i+1,j}$ for each $i < p-1$. $\square$

Again rearranging the original representation of f_{p-1},

$$(1 - c_n x_n g_0 - c_0 g_{p-1}) f_{p-1} = \sum_{i \leq p-2} (c_0 g_i + c_n g_{i+1}) f_i + \sum_{i,j} h_{i,j} \cdot x_j f_i. \tag{†}$$

Since each f_i are polynomials only in the variables $x_1, \ldots, x_{n-1}$, the condition $(\star)$ implies that $x_n^a f_i \notin x_n^a \langle f_0, \ldots, f_{i-1} \rangle + x_n^a \cdot \mathfrak{p} \cdot \tau(S, f^{\frac{1}{p}})$ for any $a > 0$. This allows one to consider (†) and the equations that follow filtered by their x_n-degree. Indeed, suppose $1 - c_n x_n g_0 - c_0 g_{p-1} \neq 0$ and consider the smallest x_n-degree term of $1 - c_n x_n g_0 - c_0 g_{p-1}$, and write it as $C x_n^a$ with $C \in K^\times$. Then considering the whole of equation (†) at x_n degree a,

$$C f_{p-1} = C_0 f_0 + \ldots + C_{p-2} f_{p-2} + \sum_{i,j} h'_{i,j} x_j f_i.$$

with $C_i \in K$, and $h'_{i,j} \in K[x_1, \ldots, x_{n-1}]$ is the x_n-degree a coefficient of $h_{i,j}$. However, $(\star)$ implies directly that this is not possible since the element on the right is an element of

$$\langle f_0, \ldots, f_{i-1} \rangle + \mathfrak{p} \cdot \tau(S, f^{\frac{1}{p}}) = \langle f_0, \ldots, f_{i-1}, x_j f_i, \ldots, x_j f_{p-1} \rangle$$

providing a contradiction to the fact that $C \neq 0$, which is to say that there is no lowest x_n-degree coefficient of $1 - c_n x_n g_0 - c_0 g_{p-1}$. Thus, $g_{p-1} = c_0^{-1}(1 - c_n x_n g_0)$.

Claim 3.6. *The condition $(\star)$ implies that $c_n g_i = -c_0 g_{i-1}$ for each $i = 1, \ldots, p-1$.*

Proof. The claim is proved by descending induction on i, starting with the case $i = p-1$. Rearranging the presentation above again using the fact that $1 - c_n x_n g_0 - c_0 g_{p-1} = 0$ yields

$$-(c_n g_{p-1} + c_0 g_{p-2}) f_{p-2} = \sum_{i \leq p-3} (c_0 g_i + c_n g_{i+1}) f_i + \sum_{i,j} h_{i,j} \cdot x_j f_i.$$

The same technique used on (†) above, by considering the lowest x_n-degree piece, implies that $c_n g_{p-1} + c_0 g_{p-2} = 0$. Now, assume $k \geq 1$ and the claim is true for $k+1, \ldots, p-1$. Then

$$-(c_n g_{k-1} + c_0 g_k) f_k = \sum_{i \le k-1} (c_0 g_i + c_n g_{i+1}) f_i + \sum_{i,j} h_{i,j} \cdot x_j f_i.$$

The same argument used in (†) again shows that $c_n g_{k+1} + c_0 g_k = 0$, or equivalently $g_{k+1} = -c_n^{-1} c_0 g_k$. □

Combining all of the data provided by Theorem 3.6, one sees that

$$c_0^{-1}(1 - c_n x_n g_0) = g_{p-1} = (-c_0 c_n^{-1}) g_{p-2} = (-c_0 c_n^{-1})^2 g_{p-3} = \ldots = (-c_0 c_n^{-1})^{p-1} g_0. \quad (††)$$

On one hand, this implies in particular that no g_i is 0 as that would imply that all g_i were 0 and the left-most side of (††) would read $c_0^{-1} = 0$. On the other hand, (††) is impossible since the x_n-degree of the left-hand side of (††) is exactly one larger than that of the right-hand side. Thus no such presentation can exist, and

$$g_{p-1} \in \tau(S, f^{\frac{1}{p}}) \setminus \tau(S, (l \cdot f)^{\frac{1}{p}}).$$

This completes the proof. □

Remark 3.7. *This proof can easily be extended to the case where K is any F-finite infinite field and $f \in F^e_* S$, and the conclusion being that for an open set of hyperplanes $H = V(l)$, $\tau(\mathbb{A}^n, f^{\frac{1}{p^e}}) \neq \tau(\mathbb{A}^n, (l \cdot f)^{\frac{1}{p^e}})$, using the same proof. Namely, choose*

$$f = \sum_{i,j} f_{i,j} \cdot F^e_*(u_j x_1 \cdots x_{n-1})^{p^e - 1} x_n^i$$

meeting the assumption

$$f_{i,j} \notin \langle f_{0,k}, \ldots, f_{i-1,k}, f_{i,1}, \ldots, \hat{f}_{i,j}, \ldots, f_{i,m} \rangle + \mathfrak{p} \cdot \tau(S, f^{\frac{1}{p^e}})$$

here, $\mathfrak{p} = \langle x_1, \ldots, x_{n-1} \rangle$, $i = 0, \ldots, p^e - 1$, and $j, k = 1, \ldots, m$ represent the basis of $F_* K$ over K. Thus in dimension greater than 2, this provides examples of pairs with the property that for a given $t > 0$, $\tau(S, f^{\frac{1}{p^e}}) \neq \tau(S, l^t \cdot f^{\frac{1}{p^e}})$, simply by taking $e \gg 0$ such that $\frac{1}{p^e} \le t$.

This gives a somewhat algorithmic way to produce counterexamples in dimension greater than 2. Indeed, once the dimension is 3 or larger, one can always find f_i meeting the condition ($\star$) in the theorem.

Additionally, one can homogenize the equation for f with respect to an additional variable x_0 and produce a more geometric counterexample:

Corollary 3.8. *Given any $\epsilon > 0$ and $n > 2$, there exist pairs $(\mathbb{P}^n_K, \Delta)$ such that for an open set $U \in (\mathbb{P}^n_K)^\vee$ and all $H \in U$,*

$$\tau(\mathbb{P}^n_K, \Delta) \neq \tau(\mathbb{P}^n_K, \Delta + \epsilon H).$$

The only new data here is that one can find such a choice of f, homogenize, and take the corresponding divisor. This is shown in the subsequent section. In addition, by considering the Veronese embedding of the examples provided above, we can conclude the following:

Corollary 3.9. *Given any $\epsilon > 0$, d a positive integer, and $n > 2$, there exist pairs $(\mathbb{P}^n_K, \Delta)$ such that for an open set U of the hypersurface of degree d in $\mathbb{P}^n$, and all hypersurfaces $H \in U$,*

$$\tau(\mathbb{P}^n_K, \Delta) \neq \tau(\mathbb{P}^n_K, \Delta + \epsilon H).$$

It is quite natural to guess that if the lowest degree hypersurfaces fail such a restriction theorem, then higher degree hypersurfaces are bound to fail spectacularly. This theorem formalizes such intuition.

3.2.2 Examples

Utilizing Theorem 3.4, one can easily produce explicit counterexamples.

Corollary 3.10 (Dimension 4)**.** *Let $K = K^p$ be an infinite perfect field of characteristic $p > 0$, and let $S = K[x, y, z, w]$. Then there exist $f \in S$ for which an open subset of hyperplanes $H = V(l)$ has the property*

$$\tau(\mathbb{A}^4, f^{\frac{1}{p}}) \neq \tau(\mathbb{A}^4, (l \cdot f)^{\frac{1}{p}})$$

Proof. Let $F_* f = F_*(xyz)^{p-1}\left[xF_*w^{p-1} + y^{p-1}zF_*w^{p-2} + y^{p-2}z^2F_*w^{p-3} + \ldots + yz^{p-1}\right]$.

It immediately meets the conditions of the theorem, by considering either x, y, or z degrees separately:

$$x \notin \langle y^{p-1}z, y^{p-2}z^2, \ldots, yz^{p-1}, x^2, xy, xz\rangle$$

$$y^{p-1}z \notin \langle y^{p-2}z^2, \ldots, yz^{p-1}\rangle + \langle y^p z, x^2, xy, xz\rangle$$

$$\vdots$$

$$yz^{p-1} \notin \langle x, y, z\rangle \cdot \langle x, y^{p-1}z, y^{p-2}z^2, \ldots, yz^{p-1}\rangle.$$

Thus for any $l = c_0 + c_x x + c_y y + c_z z + c_w w$ with no $c_\cdot = 0$, one concludes directly that $\tau(\mathbb{A}^4, f^{\frac{1}{p}}) \neq \tau(\mathbb{A}^4, (f \cdot l)^{\frac{1}{p}})$. Just for comparison's sake, in this case, we are comparing $\langle x, y^{p-1}z, y^{p-2}z^2, \ldots, yz^{p-1}\rangle$ with

$$\langle c_0 x + c_w y^{p-1}z, c_0 y^{p-1}z + c_w y^{p-2}z^2, \ldots, c_0 y^2 z^{p-2} + c_w yz^{p-1}, c_0 yz^{p-1} + c_w wx,$$

$$x^2, xy^{p-1}z, xy^{p-2}z^2, \ldots, xyz^{p-1}, y^p z, y^{p-1}z^2, \ldots, y^2 z^{p-1}, yz^p\rangle.$$

From the monomials, it is clear that these generators cannot yield a xw^j nor $y^i z^{p-i} w^j$ for any $j > 0$ or $i = 1, \ldots, p-1$. Thus, the hyperplanes that work are a subset of the closed subset of hyperplanes with at least one coefficient 0. To be more precise, c_w must be 0 and $c_0 \neq 0$ for the two ideals to agree. □

Corollary 3.11 (Dimension $n \geq 3$)**.** *Consider $S = K[x_1, \ldots, x_n]$ with K infinite perfect of characteristic $p > 0$, and $n \geq 3$. Let $H_0, \ldots, H_{p-1} \subseteq \mathbb{A}^{n-1}$ be general hyperplanes through the origin, viewed as $V(x_n) \subset \mathbb{A}^n$ with $H_i = V(l_i)$ (thus $l_i = c_{i,1}x_1 + \ldots c_{i,n-1}x_{n-1}$ for some $c_{i,j} \in K$). Consider f_i the product of all but the i^{th} of these hyperplanes:*

$$f_i = \prod_{j=0,\ldots,\hat{i},\ldots,p-1} l_j.$$

Then these f_i satisfy the conditions of the theorem. Thus $F_*f = F_*(x_1 \cdots x_{n-1})^{p-1} \sum f_i F_* x_n^i$ yields a n-dimensional counterexample in any positive characteristic to Bertini for test ideals.

Proof. As each f_i is homogeneous of degree $p-1$, the condition $(\star)$ is equivalent to

$$f_i \notin \langle f_0, \ldots, f_{i-1}\rangle.$$

This is arranged by construction, since

$$V(\langle f_0, \ldots, f_{i-1}\rangle) = \bigcap_{j=0,\ldots,i-1} V(f_j) = \bigcap_{j=0,\ldots,i-1} \left(\bigcup_{k \neq j} H_k \right) = H_i \cup \ldots \cup H_{p-1}$$

In particular, it contains H_i which $V(f_i)$ does not, so $V(f_i) \not\supseteq V(\langle f_0, \ldots, f_{i-1}\rangle)$. Therefore, $(\star)$ is met (all ideals involved are radical ideals), and one can conclude that for a general choice of hyperplane $H = V(l)$,

$$\tau(\mathbb{A}^n, f^{\frac{1}{p}}) \neq \tau(\mathbb{A}^n, (l \cdot f)^{\frac{1}{p}}).$$

□

3.3 Surfaces

I now analyze the situation for 2-dimensional normal schemes, and give a positive answer.

Theorem 3.12. *Let $X \subseteq \mathbb{P}^n$ be a normal projective surface over an infinite perfect field K of characteristic $p > 0$, and let $\Delta \geq 0$ be an effective $\mathbb{Q}$-divisor, with $K_X + \Delta$ $\mathbb{Q}$-Cartier. Then for a general choice of hyperplane $H \in (\mathbb{P}^n_K)^\vee$,*

$$\tau(X, \Delta) \otimes_{\mathscr{O}_X} \mathscr{O}_H = \tau(X \cap H, \Delta|_H).$$

Furthermore, for every $0 \leq \epsilon < 1$,

$$\tau(X, \Delta) = \tau(X, \Delta + \epsilon H).$$

Proof. Let Σ be the locus containing singularities of X and singularities of Δ. Then by normality, Σ is a finite set of closed points. Take H any hyperplane not intersecting Σ, with $X \cap H$ regular. Then by completing the local ring at a point of $x \in \Delta \cap H$, by regularity, Cohen structure theorem implies $\hat{\mathscr{O}_{X,x}} = K[\![x, y]\!]$, and by a linear change of coordinates, we can assume $H = V(y)$ and $\Delta = t \cdot V(x)$ for some positive rational number t. In this complete local setting, $\hat{\mathscr{O}_H} = K[\![x]\!]$, and $\Delta|_H$ is simply $t \cdot V(x)$. Finally, the following computations follow directly from Section 3.1:

$$\tau(K[\![x, y]\!], x^t) = \sum_{e \geq 0} \Phi^e_S(F^e_* \langle x^{\lceil t(p^e-1) \rceil} \rangle) = \langle x^{\lfloor \frac{\lceil t(p^e-1) \rceil}{p^e} \rfloor} \rangle_{e \geq 0} = \langle x^{\lceil t \rceil - 1} \rangle$$

and thus

$$\tau(K[\![x, y]\!], x^t) \otimes_{K[\![x,y]\!]} K[\![x]\!] = \tau(K[\![x, y]\!], x^t) \cdot K[\![x]\!] = \langle x^{\lceil t \rceil - 1} \rangle = \tau(K[\![x]\!], x^t).$$

Finally, since every point of $X \cap H$ and $\Delta \cap H$ is smooth in X, Δ respectively, the test ideal sheaves agree as well.

The second statement follows by the same logic, as $\tau(X, \Delta + \epsilon H) = \langle x^{\lceil t \rceil - 1} \rangle$ in $K[\![x, y]\!]$. □

3.4 Test Ideal and Restriction to Hypersurfaces

In this section, it is demonstrated that the more common incantation of Bertini theorems is also false.

3.4.1 Counterexamples to Bertini

In addition, I show that the direct analogue of Bertini's Second Theorem for test ideals fails in general. Namely, there exists (X, Δ) with the property that for a general hyperplane H, one has

$$\tau(X, \Delta) \otimes_{\mathscr{O}_X} \mathscr{O}_H \neq \tau(H, \Delta|_H).$$

Definition 3.13. *[3, Definition 3.12] Let S be a Noetherian ring over an F-finite infinite field K. For an ideal I of S, let $\tau(S, \not\subseteq I, \Delta)$ be the smallest ideal J of S with the property that $J \not\subseteq I$, and such that J is compatible with all $\varphi \in \mathrm{Hom}_S(F^e_* S(\lceil (p^e - 1)\Delta \rceil), S)$.*

This slight modification to the definition of the test ideal gives the correct object to study, in the following sense:

Lemma 3.14. *Given (X, Δ) as above, the containments*

$$\tau(X, \Delta) \supseteq \tau(X, \Delta + \frac{1}{p}H) \supseteq \tau(X, \not\subseteq I_H, \Delta)$$

hold. Moreover, in the case considered throughout the paper, the restriction of $\tau(X, \not\subseteq I_H, \Delta) + \langle l \rangle$ to $H = V(l)$ is $\tau(S/\langle l \rangle, \Delta|_H)$.

Proof. The first containment is apparent by the discussion in Section 3.1. Furthermore, one has the containment

$$\tau(X, \not\subseteq I_H, \Delta) \subseteq \sum_{e \geq 0} \Phi^e_S(F^e_* f^{\lceil \frac{p^e-1}{p^e} \rceil} l^{p^e-1} c) = \sum_{e \geq 0} \tau(S, f^{\frac{1}{p^e}} l^{\frac{p^e-1}{p^e}}).$$

Here c is a generalized test element. Since S is Noetherian, the containment $\tau(X, \not\subseteq I_H, \Delta + H) \subseteq \tau(S, f^{\frac{1}{p^e}} l^{\frac{p^e-1}{p^e}})$ holds for some $e \gg 0$. The second statement of the Lemma is worked out in even broader generality as [3, Theorem 3.15]. □

There is a minor obstruction in going directly from Theorem 3.4 to a conclusion about restriction to hypersurface. In particular, it isn't clear how to ensure that even when $\tau(S, f^t) \neq \tau(S, (f \cdot l)^t)$, that upon restriction to $S/\langle l \rangle$, the two ideals are not equal. However, it is still possible to give a general class of examples for which $\tau(S, f^t)|_H \neq \tau(S/\langle l \rangle, \bar{f}^t)$.

Theorem 3.15. *Let $S = K[x_1, \ldots, x_n]$, for K a perfect infinite field and let*

$$f = \sum_{i=0}^{p^e-1} f_i F_*^e x_1^{p^e-1} \cdots x_{n-1}^{p^e-1} x_n^i$$

with each $f_i \in K[x_1, \ldots, x_{n-1}]$, homogeneous of the same degree d, and such that the f_i span a K-vectorspace of S_d of dimension at least 2. Then for an open collection of hyperplanes H,

$$\tau(S, f^{\frac{1}{p}}) \cdot R/\langle l \rangle \neq \tau(S/\langle l \rangle, \bar{f}^{\frac{1}{p}})$$

Proof. Let

$$l = c_0^p + c_1^p x_1 + \ldots + c_n^p x_n.$$

$\tau(S, f^{\frac{1}{p}}) = \langle f_i \rangle$, as usual. Assume that $c_n \neq 0$ in the presentation of l. Then $S/\langle l \rangle \cong K[x_1, \ldots, x_{n-1}]$ via

$$x_i \mapsto x_i \quad i = 1, \ldots, n-1$$

$$x_n \mapsto -c_n^{-p}(c_0^p + c_1^p x_1 \ldots + c_{n-1}^p x_{n-1}).$$

Note that $\bar{f}_i$ are also homogeneous of degree d using the standard grading of $K[x_1, \ldots, x_{n-1}]$. Moreover, one can compute $\tau(S/\langle l \rangle, \bar{f}^{\frac{1}{p}})$ with respect to any basis of $F_* R$, such as $F_* x_1^{\alpha_1} \cdots x_{n-1}^{\alpha_{n-1}}$, where $0 \leq \alpha_j \leq p-1$. So

$$\begin{aligned} F_*\overline{f} &= \sum_{i=0}^{p-1} f_i F_*^e x_1^{p-1} \cdots x_{n-1}^{p-1} x_n^i \\ &= \sum_{i=0}^{p-1} f_i F_*^e x_1^{p-1} \cdots x_{n-1}^{p-1} \cdot (-c_n)^{-ip} (c_0^p + c_1^p x_1 + \ldots + c_{n-1}^p x_{n-1})^i. \end{aligned}$$

Therefore, $\bar{f}$ expressed in this basis has non-zero coefficients associated to $F_* \boldsymbol{x}^{\boldsymbol{p-1+\theta}}$, where exponents are taken $\pmod p$ and $0 \leq |\boldsymbol{\theta}| \leq p-1$. Applying trace of $S/\langle l \rangle$, the generators come in two types:

- $\sum_{i=0}^{p-1} (-c_0 c_n^{-1})^i \cdot f_i$, the coefficient of $F_* \boldsymbol{x}^{\boldsymbol{p-1}} = F_* x_1^{p-1} \cdots x_{n-1}^{p-1}$.
- $\boldsymbol{x}^{\lceil \frac{\boldsymbol{\theta}}{p} \rceil} \sum_{i=|\boldsymbol{\theta}|}^{p-1} d_{\boldsymbol{\theta},i} \cdot f_i$ the coefficient of $F_* \boldsymbol{x}^{\boldsymbol{p-1+\theta}}$ ($\pmod p$ exponents)

where $d_{\boldsymbol{\theta},i} = c_0^{i-|\boldsymbol{\theta}|}(-c_n)^{-i}\boldsymbol{c}^{\boldsymbol{\theta}} \in K$. It is clear that $\tau(S/\langle l\rangle, \bar{f}^{\frac{1}{p}})$ is homogeneous and

$$\tau(S/\langle l\rangle, \bar{f}^{\frac{1}{p}}) \subseteq \langle \sum_i (c_0 c_n^{-1})^i f_i\rangle + \mathfrak{p}\tau(S, f^{\frac{1}{p}}) \cdot S/\langle l\rangle$$

where $\mathfrak{p} = \langle x_1, \ldots, x_{n-1}\rangle$. Intersecting $\tau(S/\langle l\rangle, \bar{f}^{\frac{1}{p}})$ with $(S/\langle l\rangle)_d$ yields a 1-dimensional vector space $\langle \sum_i (c_0 c_n^{-1})^i f_i\rangle_K$. Therefore, since both ideals are homogeneous, one can compare them degree-wise, and note that

$$\left(\tau(S, f^{\frac{1}{p}}) \cdot S/\langle l\rangle\right)_d \neq \tau(S/\langle l\rangle, \bar{f}^{\frac{1}{p}})_d.$$

and therefore, the two ideals are distinct. This completes the proof. □

If we further assume that the f_i meet the condition $(\star)$ above, we can find a simultaneous counterexample to restriction and Bertini for test ideals.

3.4.2 Further Examples

Now, to provide a proper counterexample, it suffices to show that $\tau(S,\Delta)/\tau(S,\Delta + \frac{1}{p}H)$ is non-zero in $S/\langle l\rangle$. An easy resolution can be demonstrated in fixed characteristic:

Example 3.16. *Let K be an infinite perfect field of characteristic 3, and let $S = K[x,y,z]$. Let*

$$f = x^5y^2z^2 + y^5x^2z$$

so that

$$F_*f = xF_*x^2y^2z^2 + yF_*x^2y^2z.$$

Then by Theorem 3.15, there is an open dense set of hyperplanes $H = V(l)$ such that

$$\tau(S, f^{\frac{1}{p}}) \cdot S/\langle l\rangle \neq \tau(S/\langle l\rangle, \bar{f}^{\frac{1}{p}}).$$

Letting $l = c_1^p + c_x^p x + c_y^p y - c_z^p z$, one can actually see this directly, assuming $c_z \neq 0$:

$$\begin{aligned} F_*fl =& (c_z^{-2}c_1^2 x + c_z^{-1}c_1 y)F_*x^2y^2 + (c_1c_yc_z^{-2}xy + c_yc_z^{-1}y^2)F_*x^2 + (c_1c_xc_z^{-2}x^2 + c_xc_z^{-1}xy)F_*y^2 \\ &+ (c_x^2c_z^{-1}x^2)F_*xy^2 + (c_y^2c_z^{-1}xy)F_*x^2y + (c_xc_yc_z^{-1}x^2y)F_*1. \end{aligned}$$

And as a result,

$$\tau(S, f^{\frac{1}{p}}) \cdot S/\langle l\rangle = \langle x,y\rangle \neq \langle c_1x + c_zy, x^2, xy, y^2\rangle \supseteq \tau(S/\langle l\rangle, \bar{f}^{\frac{1}{p}}).$$

Naturally, since they don't agree in degree 1 (identifying $S/\langle l\rangle$ with $K[x,y]$), they differ.

Additionally, it can be seen that the examples of Section 3.2.2 also provide counterexamples to Bertini.

Proposition 3.17. *(Dimension 4) Consider Corollary 3.10. Let Δ be the divisor defined associated to f. Then for an open dense subset of $H = V(l)$,*

$$\tau(\mathbb{P}^4, \frac{1}{p}\Delta)|_H \neq \tau(H, \frac{1}{p}\Delta|_H).$$

Proof. Note that in the proof of Proposition 3.4, it is shown that each $f_i \notin \tau(S, (lf)^{\frac{1}{p}})$, where

$$f = f_{p-1} \cdot F_*(x_1 \cdots x_{n-1})^{p-1} x_n^{p-1} + f_{p-2} \cdot F_*(x_1 \cdots x_{n-1})^{p-1} x_n^{p-2} + \ldots + f_0 \cdot F_*(x_1 \cdots x_{n-1})^{p-1}.$$

Thus, given $l = c_0 + c_x x + c_y y + c_z z + c_w w$ with each $c_i \neq 0$, the proof realizes the fact that

$$x, y^{p-1}z, y^{p-2}z^2, \ldots, yz^{p-1} \notin \tau(S, (l \cdot f)^{\frac{1}{p}}).$$

In particular, they are not in $\tau(S, \not\subseteq I_H, (l \cdot f)^{\frac{1}{p}})$. Now, consider the image of $\tau(S, (l \cdot f)^{\frac{1}{p}}) + \langle l \rangle$ in $S/\langle l \rangle$, and call it J for simplicity.

Begin by eliminating w from all equations by virtue of the relation $w = -c_w^{-1}(c_0 + c_x x + c_y y + c_z z)$. Then

$$f|_H = \left[(c_w^{-1} c_0)^{\frac{p-1}{p}} x + (c_w^{-1} c_0)^{\frac{p-2}{p}} y^{p-1} z + \ldots + yz^{p-1} \right] F_*(xyz)^{p-1} + \sum_{\boldsymbol{\alpha} < \boldsymbol{p-1}} s_{\boldsymbol{\alpha}} F_* x^{\boldsymbol{\alpha}}.$$

Now, utilizing the fact that the test ideal can be computed as in Proposition 3.1, but with respect to any basis of $F_*(S/\langle l \rangle)$ over $S/\langle l \rangle$, such as the $(xyz)^{\boldsymbol{\alpha}}$ with $\boldsymbol{0} \leq \boldsymbol{\alpha} < \boldsymbol{p}$ basis, it is easily seen that

$$\tau(H, f|_H^{\frac{1}{p}}) = \langle (c_w^{-1} c_0)^{\frac{p-1}{p}} x + (c_w^{-1} c_0)^{\frac{p-2}{p}} y^{p-1} z + \ldots + yz^{p-1}, s_{\boldsymbol{\alpha}} \rangle.$$

It is also easy to see that each $s_{\boldsymbol{\alpha}} \in \langle x, y, z \rangle \cdot \left(\tau(S, f^{\frac{1}{p}}) + \langle l \rangle \right)$, since the replacement of w with $-c_w^{-1}(c_0 + c_x x + c_y y + c_z z)$ yields either a constant coefficient (accounted for in the explicitly written down generator) or some higher multiple of x, y, or z, and further, every term of the original f had $F_*(xyz)^{p-1}$ in it already. Thus it is clear that $0 \neq x \notin \tau(H, f|_H^{\frac{1}{p}})$, whereas $x \in \tau(X, f^{\frac{1}{p}})$. □

Proposition 3.18. *(Dimension $n \geq 3$) Consider Corollary 3.11. If Δ is the divisor associated to f, then for an open dense subset of $H = V(l)$,*

$$\tau(\mathbb{P}^n, \frac{1}{p}\Delta)|_H \neq \tau(H, \frac{1}{p}\Delta|_H).$$

Proof. Restrict the condition on $H = V(l)$ to be as in Theorem 3.4 with the additional property that H is not one of the chosen general H_i of Corollary 3.11, to avoid the case where all but one (or even any) of the f_i are zero upon restriction to H. Let $R = S/\langle l \rangle$. Following similar techniques to the proof of Proposition 3.17, one concludes that

$$\tau(S, f^{\frac{1}{p}})|_R = \langle \bar{f}_0, \ldots, \bar{f}_{p-1} \rangle$$

$$\tau(R, \bar{f}^{\frac{1}{p}}) = \langle \sum_{i=0}^{p-1} (c_0 c_w)^{\frac{i}{p}} \bar{f}_i, s_{\alpha} \rangle.$$

where s_{α} are homogeneous elements with $\deg(s_{\alpha}) \geq p$. Therefore, it is clear that since all of the $\bar{f}_i$ are distinct and non-zero elements of degree $p-1$ in R, not all of them can be in $\tau(R, \bar{f}^{\frac{1}{p}})$.

This further extends to the case of hypersurfaces in general position (having the expected intersection dimension). □

This answers [16, Question 8.3] in the negative in any dimension larger than 2.

3.5 Application to Flat Families

A similar argument provides a counterexample to a natural extension of a question of Hochster and Huneke.

Theorem 3.19. *[13, Theorem 7.3(c)] Let $\varphi : R \to S$ be a flat homomorphism of Noetherian rings in characteristic $p > 0$. Suppose that R is F-regular, S is excellent, and that φ has regular fibers. Then S is also F-regular.*

Note that here a map φ of rings (or schemes) is said to have regular fibers if for every prime $\mathfrak{p} \in \operatorname{Spec}(R)$,

$$S \otimes_R \kappa(\mathfrak{p}) = S_{\mathfrak{p}}/\mathfrak{p}S_{\mathfrak{p}}$$

is Noetherian and geometrically regular over $\kappa(\mathfrak{p})$.

This brings about the following question:

Question 3.20. *If $\pi : X \to Y$ is a flat morphism of characteristic $p > 0$ Noetherian schemes with Y assumed $\mathbb{Q}$-Gorestein, and π having regular fibers, with Δ a $\mathbb{Q}$-divisor on Y. Is it true that $\tau(Y, \Delta) \cdot \mathscr{O}_X = \tau(X, \pi^* \Delta)$?*

The $\mathbb{Q}$-Gorenstein hypothesis allows one to make sense of $\pi^*\Delta$. The result of Hochster and Huneke holds for strongly F-regular pairs, where both test ideals are their respective sheaves of rings. In addition, K. Smith and A. Bravo proved a theorem in this direction assuming that the morphism is smooth:

Theorem 3.21. *[4, Theorem 5.1] Let $f : R \to S$ be a smooth (locally of finite type) homomorphism of reduced excellent rings of characteristic $p > 0$. Assume $R_{\mathfrak{m}}/\mathfrak{m}R_{\mathfrak{m}}$ is a perfect field for each maximal ideal $\mathfrak{m}$ of R. Then $\tau_R(S) = \tau_S$.*

In proving this theorem, they also prove one containment more generally:

Theorem 3.22. *[4, Proposition 3.1] Let $f : R \to S$ be a homomorphism of Noetherian reduced rings of characteristic $p > 0$, and assume R is essentially of finite type over an excellent ring or that R is F-finite. Suppose that S is a free R module. Then $\tau_S \subseteq \tau_R \cdot S$.*

However, Proposition 3.17 (or similarly Proposition 3.18) can be used to show that this question is false more generally.

Theorem 3.23. *Consider Z to be the reduce closed subscheme of $\mathbb{P}^n \times (\mathbb{P}^n)^\vee$ determined by taking the closure of the set of pairs (P, H) of points P and hyperplanes H through the point P. Then $Z \to \mathbb{P}^n$ is a flat morphism, with regular fibers, such that there exists Δ a $\mathbb{Q}$-divisor on $\mathbb{P}^n$ with*

$$\tau(\mathbb{P}^n, \Delta) \cdot \mathscr{O}_Z \not\subseteq \tau(Z, \pi^*\Delta).$$

Proof. The statement that $Z \to \mathbb{P}^n$ is a flat morphism is proved in the course of [16, Theorem 3.7]. For simplicity, and to be cohesive with Section 3.1 and Corollary 3.10, everything is computed (locally) in $\mathbb{A}^n$. Thus the morphism being considered locally corresponds to the ring morphism

$$R = K[x_1, \ldots, x_n] \to K[x_1, \ldots, x_n](y_0^{\frac{1}{p}}, y_1^{\frac{1}{p}}, \ldots, y_n^{\frac{1}{p}})/\langle y_0 + y_1x_1 + \ldots + y_nx_n\rangle = S.$$

Here the new variables y_i should be thought of as the coefficients of the linear form l as above. The fibers of this morphism are regular (polynomial) rings. It only remains to compute the corresponding test ideals. Let Φ_S be the trace map described at the beginning of this section. Then a generator of $\mathrm{Hom}_R(F_*R, R)$ as an F_*R module can be given as $\Phi_R = \Phi_S(F_*(l^{p-1} \cdot -))$, where $l = y_0 + y_1x_1 + \ldots + y_nx_n$.

Consider f as in Proposition 3.17. For a general choice of l (with coefficients), it was shown that

$$\Phi_S(\langle F_* f\rangle) \neq \Phi_S(\langle F_*(l \cdot f)\rangle) \supseteq \Phi_S(\langle F_*(l^2 \cdot f)\rangle) \supseteq \ldots \supseteq \Phi_S(\langle F_*(l^{p-1} \cdot f)\rangle) = \Phi_R(\langle F_* f\rangle).$$

Thus the two test ideals stated in the theorem cannot agree, as was to be shown. $\square$

CHAPTER 4

REMAINING QUESTIONS

In this section, I discuss open questions which stem from the above results.

4.1 General Criterion for Bertini

The previous section implies that an analog of Bertini's Theorem for test ideals is not possible for pairs (X, Δ) in general, or even restricting our attention to irreducible normal schemes. In fact, based on the information presented above, it seems probable that any projective scheme X of dimension greater than or equal to 3 has $\mathbb{Q}$-divisors Δ for which the restriction theorem is false, since on the regular locus, it seems to fail spectacularly.

On the other hand, there are also classes (X, Δ) for which such a theorem is possible. For example, if f is of the form $f_\alpha F_* x^\alpha$, or if the $F_* x^\alpha$ are in sufficiently low degrees, the restriction theorem holds. More generally, in many cases where the f_i are not quite as independent as condition $(\star)$ enforces, we can still conclude that a general hypersurface will have the restricted test ideal. Another general class comes from Theorem 2.6, which states that for Sharply F-pure or Strongly F-regular pairs (X, Δ), an analogue of the second theorem of Bertini in fact holds. In the case of Strongly F-regular pairs, this implies that $\tau(X, \Delta) = \mathcal{O}_X$ is a very rigid condition.

So a general question one could pose would be which pairs (X, Δ) have such a property? Or more specifically, is there some geometric or arithmetic property that is governing whether or not the theorem holds? The most likely candidate seems to be that locally, the extension of residue fields $k_X(p)/k_H(p)$ is highly inseparable. However, not much is known in this direction.

4.2 Bounding Degrees

One particular thing to note is that Corollary 3.11 has Δ of a fairly large degree; $(p + n) \cdot (p-1)$ where n is the dimension, or $O(p^2)$. It is possible that the theorem holds up to a particular degree for a given dimension of variety n and characteristic p, which would ensure at least that the f_i don't meet the condition $(\star)$.

One could also ask if there is some type of locus determined by the test ideal for which a hyperplane not intersecting the said locus is sufficient to guarantee that the restriction statement holds, and if so, what properties does this locus have? In particular, if it was a finite collection of points, a general hyperplane would always miss this loci and the Bertini theorem would hold. And if it were of a higher dimension, it is possible that one could choose hypersurfaces H of a class which intersects the locus sufficiently nicely.

4.3 Direct Counterexamples for Varieties (Non-pairs)

A very important question that remains open is whether or not one can use this type of technique to find counterexamples to the original Hochster-Huneke question. Namely, could it still be true when considering a variety instead of pairs?

Question 4.1. *For an irreducible projective variety X, and a very ample linear series $|D|$, is it true that for a general choice of $\Delta \in |D|$, one has that $\tau(X) = \tau(\Delta \cap X)$?*

The answer is very likely no based on the findings of this paper, but an explicit counterexample is still unknown to the author.

In particular, the class of f provided from Theorem 3.4, one could consider $X = V(f)$. In this case, the generating morphism has a particularly nice form $\Phi_X(g) = \Phi(gF_* f^{p-1})$, for which we have already shown that the desired restriction theorem fails. However, these examples are all reducible, as one can simply factor out $x_i^{p^e-1}$ for $i = 1, 2, \ldots, n-1$, so they cannot serve as direct counterexamples to Question 4.1.

4.4 F-rationality

Two geometric properties of interest in positive characteristic algebraic geometry and commutative algebra, which are related to the test ideal, are F-injectivity and F-rationality. They are defined as follows:

Definition 4.2. *An F-finite local ring $(R, \mathfrak{m})$ is said to be F-injective if the induced map $H^i_{\mathfrak{m}}(R) \to H^i_{\mathfrak{m}}(F^e_* R)$ is injective for every $i \geq 0$. If we assume R has a dualizing complex, then we can equivalently state that R is F-injective if $\mathbf{h}^i(F^e_* \omega^{\cdot}_R) \to \mathbf{h}^i(\omega^{\cdot}_R)$ is surjective for every i. A variety* *is said to be F-injective if every localization at a maximal ideal is F-injective.*

This condition is the characteristic p-analogue of being DuBois in characteristic 0, and is a weakening of F-purity since local cohomology splits across direct sums. The direct analog of the second theorem of Bertini for F-injectivity is known to be false in the case where R is allowed to be non-normal: [16, Proposition 7.4]. It is however still open in the case of normal F-injective varieties, as the counter-example used the weakly normal but not WN1 construction (referenced in [7]).

Definition 4.3. *A local Cohen-Macaulay ring R is said to be F-rational if the canonical module ω_R is a simple module under the action of the trace $\Phi_S : F^e_* \omega_R \to \omega_R$. That is to say that ω_R is the smallest non-zero Φ_S-compatible submodule.*

This is the characteristic $p > 0$ analogue of being rational. It is currently unknown whether a Bertini-type theorem holds true for F-rational singularities even in the non-normal case. To answer such a question in the negative, one could find an F-rational ring R with $\omega_R \cong I \subset R$, and I meets the criteria of Theorem 3.4, or a similar criteria. This would result in ω_R being Φ_R-simple, but $\omega_{R/\langle l \rangle} = \mathrm{Hom}_R(R/\langle l \rangle, \omega_R)$ not being $\Phi_{R/\langle l \rangle}$-simple, and thus F-rationality would break descending to a general hypersurface.

APPENDIX

MACAULAY2 CODE

In this section, I include some Macaulay2 code used to find the set, as well as the proportion, of hyperplanes defined over a finite field which fail the desired restriction theorem for a given ideal I.

A.1 TestForHypersurfaceBehavior.m2

```
testForElement = method();
testForElement(Ideal,ZZ) := opt (I,n) -> (

S := ring(I);
K := coefficientRing S;
p := char K;
K = GF(p^n,Variable=>b);
newS := K[first entries (vars S)];
newVars := vars newS;
dimen := numColumns newVars;
newI := sub(I,newS);
Lines := p^(n*dimen)-1;
newI2 := ethRoot(newI,1);
--ListOfFails = {};
NumOfFails = 0;
```

This section of the code creates a polynomial ring with the same parameters as the original ring, but instead of using $\mathbb{F}_p$ as its base field, it is taken over a Galois extension of the original finite field of degree n. This allows for a larger class of hyperplanes to be taken to better approximate the case of an algebraically closed, or at least infinite, field.

```
i:=1;
while(i<Lines) do (
Linei = getCoefficientListFromInteger(i,p^n,dimen);
Vect = {};

j=0;
```

```
while (j<dimen) do(
elem = Linei#j;
if (elem == 0) then ( Vect = Vect | {sub(0,newS)}; )
else ( Vect = Vect | {sub(b^(elem-1),newS)}; );
j= j+1;
);

Vect = transpose(matrix{Vect});
l = 1+ (first entries(newVars*Vect))#0;
J = newI*ideal(l^(p-1));
J = ethRoot(J,1);
if (newI2 != J) then (
--ListOfFails = {ListOfFails = ListOfFails | {l};};
NumOfFails = NumOfFails + 1;
);
i = i+1;
);
NumOfFails/Lines
);

testForElement(RingElement,ZZ) := (F,n) -> (
testForElement(ideal(F),n)
);
```

Here, the while loop runs through all of the hyperplanes defined over $\mathbb{F}_{p^n}$ with non-zero constant coefficient. It does this by taking the p-adic expansion of a number i and taking its coefficients under this expansion. Then it takes a primitive b of the Galois field, and raises b to this power, adjoining it as the coefficient of a linear form. This is the object l. Then it computes and compares $\Phi_S(F^e_* J) = J^{\left[\frac{1}{p}\right]} = ethroot(I,1) = newI2$ with $\Phi_S(F^e_* J \cdot l^{p-1})$. If these two ideals are non-equal, it adds 1 to the count of failures of restriction. The option is also there to remove the comments to maintain a list of lines for which the restriction fails (though this is very RAM intensive).

A.2 Some Examples Tested in Small Degree

A.2.1 Counterexamples

Example A.1. *In the following, a basic example following the procedure illustrated above in Theorem 3.4 is tested over $\mathbb{F}_3$, with coefficients in $\mathbb{F}_3$, $\mathbb{F}_9$, and $\mathbb{F}_{27}$, respectively:*

```
i20 : load "TestForHypersurfaceBehavior.m2"; R = ZZ/3[x,y,z];
      f=x^6*y^6*x^2*y^2*z^2 + x^9*x^2*y^2*z + y^9*x^2*y^2
```

```
        8 8 2     11 2      2 11
o22 = x y z  + x   y z + x y
i23 : testForElement(f,1)
      17
o23 = --
      26
i24 : testForElement(f,2)
      647
o24 = ---
      728
i25 : testForElement(f,3)
      18953
o25 = -----
      19682
```

So what the program has demonstrated is that the proportion of hyperplanes in each case which fail the restriction is approximately .654, .889, and .963. So it is tending towards 1 (an open set of hyperplanes) as we allow more coefficients into the system. This is representative of the fact that the program does not exclude lines which have coefficients of zero in some of the terms. For example, $1 + x$ *is tested when* $i = 1$*. This is done so that the program can detect examples which don't fit precisely into the form of Theorem 3.4.*

Similar to Corollary 3.10, the sufficient condition on $l = c_1 + c_x x + c_y y + c_z z$ *is that* $c_z \neq 0$ *and* $c_1 = 0$*. Since the program assumes* $c_1 \neq 0$ *and not all of* c_x, c_y*, and* c_z *are* 0 *to avoid trivialities, what these proportions are representing is*

$$\frac{|\mathbb{F}^3_{p^n}| - |\mathbb{F}^2_{p^n}| - 1}{|\mathbb{F}^3_{p^n}| - 1} = \frac{p^{3n} - p^{2n} - 1}{p^{3n} - 1}$$

Another example with a similar setup can be constructed in dimension 4:

Example A.2. *In this example, the characteristic is* 5*, and* $f_{p-1} = f_4 = x^2$*,* $f_3 = y^2$*,* $f_2 = z^2$*,* $f_1 = x * y$*,* $f_0 = y * z$*.*

```
i1 : load "TestForHypersurfaceBehavior.m2"; R = ZZ/5[x,y,z,w];
i3 : f = (x^2)^5*x^4*y^4*z^4*w^4+ (y^2)^5*x^4*y^4*z^4*w^3 +...
      14 4 4 4     4 14 4 3     4 4 14 2     9 9 4     4 9 9
o3 = x  y z w  + x y  z w  + x y z  w  + x y z w + x y z
i4 : testForElement(f,1)
     499
o4 = ---
     624
i5 : testForElement(f,2)
```

```
     374999
o5 = ------
     390624
```

Here i5 took several hours to execute.

A.2.2 Random Examples

By cherry picking the examples, it becomes easy to produce counterexamples. However, choosing examples randomly yields different results. In particular, it seems that most choices of divisor defined by f yield the expected result for restriction. I omit some of the output of Macaulay2 for brevities sake.

Example A.3.

```
i4 : load "TestForHypersurfaceBehavior.m2"; R = ZZ/3[x,y,z]; f=random(3,R)
      3    2             2       2      2    3
o6 = x  + x y - x*y*z + y z + x*z  - y*z  + z
i7 : testForElement(f,1)
o7 = 0
i8 : testForElement(f,2)
o8 = 0
i9 : testForElement(f,3)
o9 = 0
```

Example A.4. *I include the output of f for reasons of reproducibility.*

```
 R = ZZ/5[x,y,z]; f=random(10,R)
       9   7 3  6 4  5 5  4 6  3 7  2 8      9   10  9
o11 = -x y+x y +x y +x y +x y +x y -x y -2x*y +2y  +x z -
      ----------------------------------------------------------
        7 2    6 3   5 4    4 5    3 6    2 7      8   9   8 2
      2x y z-2x y z-x y z+2x y z-2x y z-2x y z+2x*y z-y z-x z
      ----------------------------------------------------------
         6 2 2  4 4 2  2 6 2     7 2   8 2   7 3  5 2 3  4 3 3
      -2x y z -x y z +x y z -2x*y z -2y z -2x z +x y z +x y z -
      ----------------------------------------------------------
        3 4 3   2 5 3     6 3  7 3   6 4  5   4  4 2 4  3 3 4
      2x y z +2x y z +2x*y z -y z +2x z +x y*z -x y z -x y z -
      ----------------------------------------------------------
        2 4 4   5 4   6 4  3 2 5  5 5   4 6  3   6   2 2 6
      2x y z -x*y z -2y z -x y z -y z +2x z +x y*z +2x y z  -
      ----------------------------------------------------------
```

```
         3 6    4 6    3 7  2    7     2 7      8   2 8    9    9
      x*y z -2y z -2x z +x y*z +2x*y z -x*y*z +2y z +x*z +y*z
      ----------------------------------------------------------
        10
      +z
o11 : R
i12 : testForElement(f,1)
o12 = 0
i13 : testForElement(f,2)
o13 = 0
```

In this case, every $l = c_0 + c_x x + c_y y + c_z z$ with a non-zero c_0 yields a test ideal which restricts

The pattern follows similarly for many randomly generated examples considered in degree 20 over $\mathbb{Z}/5$ and degree 35 over $\mathbb{Z}/7$ up to field extension degree 4. So one likely concludes that this phenomenon where the test ideal of a pair doesn't restrict to a general hyperplane is very sparse.

REFERENCES

[1] B. Bhatt and P. Scholze, *Projectivity of the witt vector affine grassmannian*.

[2] M. Blickle, M. Mustaţă, and K. E. Smith, *Discreteness and rationality of F-thresholds*, Michigan Math. J., 57 (2008), pp. 43–61. Special volume in honor of Melvin Hochster.

[3] M. Blickle, K. Schwede, S. Takagi, and W. Zhang, *Discreteness and rationality of F-jumping numbers on singular varieties*, Math. Ann., 347 (2010), pp. 917–949.

[4] A. Bravo and K. E. Smith, *Behavior of test ideals under smooth and étale homomorphisms*, J. Algebra, 247 (2002), pp. 78–94.

[5] M. Brion and S. Kumar, *Frobenius splitting methods in geometry and representation theory*, vol. 231 of Progress in Mathematics, Birkhäuser Boston, Inc., Boston, MA, 2005.

[6] A. Bydlon, *Macaulay2 code to test restriction of σ and τ*. Available at http://www.math.utah.edu/ bydlon/TestForHypersurfaceBehaviorSigma.m2.

[7] C. Cumino, S. Greco, and M. Manaresi, *Hyperplane sections of weakly normal varieties in positive characteristic*, Proc. Amer. Math. Soc., 106 (1989), pp. 37–42.

[8] R. Fedder and K. Watanabe, *A characterization of f-regularity in terms of f-purity*.

[9] D. R. Grayson and M. E. Stillman, *Macaulay2, a software system for research in algebraic geometry*. Available at http://www.math.uiuc.edu/Macaulay2/.

[10] N. Hara, *A characteristic p analog of multiplier ideals and applications*, Comm. Algebra, 33 (2005), pp. 3375–3388.

[11] N. Hara and S. Takagi, *On a generalization of test ideals*, Nagoya Math. J., 175 (2004), pp. 59–74.

[12] N. Hara and K.-I. Yoshida, *A generalization of tight closure and multiplier ideals*, Trans. Amer. Math. Soc., 355 (2003), pp. 3143–3174 (electronic).

[13] M. Hochster and C. Huneke, *F-regularity, test elements, and smooth base change*, Trans. Amer. Math. Soc., 346 (1994), pp. 1–62.

[14] R. Lazarsfeld, *Positivity in algebraic geometry. II*, vol. 49 of Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], Springer-Verlag, Berlin, 2004. Positivity for vector bundles, and multiplier ideals.

[15] V. B. Mehta and V. Srinivas, *Varieties in positive characteristic with trivial tangent bundle*, Compositio Math., 64 (1987), pp. 191–212. With an appendix by Srinivas and M. V. Nori.

[16] K. Schwede and W. Zhang, *Bertini theorems for F-singularities*, Proc. Lond. Math. Soc. (3), 107 (2013), pp. 851–874.

[17] K. E. Smith, *The multiplier ideal is a universal test ideal*, Comm. Algebra, 28 (2000), pp. 5915–5929. Special issue in honor of Robin Hartshorne.

[18] S. Takagi, *An interpretation of multiplier ideals via tight closure*, J. Algebraic Geom., 13 (2004), pp. 393–415.